New Classes of Salagean type Meromorphic Harmonic Functions

Hakan Bostancı, and Metin Öztürk

Abstract**—**In this paper, a necessary and sufficient coefficient are given for functions in a class of complex valued meromorphic harmonic univalent functions of the form $f = h + \bar{g}$ using Salagean operator. Furthermore, distortion theorems, extreme points, convolution condition and convex combinations for this family of meromorphic harmonic functions are obtained.

Keywords**—**Harmonic mappings, Meromorphic functions, Salagean operator.

I. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain D if both u and v are real harmonic in D. In any simply connected domain $D \subset \mathbf{C}$ we can write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [2]). In [3], Hengartner and Schober investigated functions harmonic in the exterior of the unit disc $\tilde{U} = \{z : |z| > 1\}$. They showed that complex valued, harmonic, sense preserving, univalent mapping f must admits the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z|,$$

where

$$h(z) = \alpha z + \sum_{k=1}^{\infty} a_k z^{-k}$$

and

$$g(z) = \beta \bar{z} + \sum_{k=1}^{\infty} b_k z^{-k}$$

for $0 \le |\beta| < |\alpha|,\ A \in \mathbf{C}$.

Let MH denote the class of functions

$$f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} a_k\, z^k + \overline{\sum_{k=1}^{\infty} b_k\, z^k} \qquad (1)$$

which are harmonic in the punctured unit disk $U\backslash\{0\}$. $h(z)$ and $g(z)$ are analytic in $U\backslash\{0\}$ and U, respectively, and $h(z)$ has a simple pole at the origin with residue 1 here.

For $f = h + \bar{g}$ given by (1), Jahangiri [4] defined the modified Salagean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)}; \quad n = 0, 1, 2, \cdots, \qquad (2)$$

where

$$D^n h(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n a_k\, z^k$$

and

$$D^n g(z) = \sum_{k=1}^{\infty} k^n b_k\, z^k.$$

A function $f(z) \in MH$ is said to be in the subclass MHS^* of meromorphically harmonic starlike in $U\backslash\{0\}$ if it satisfies the condition

$$Re\left\{-\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} > 0, \quad z \in U\backslash\{0\}.$$

Now we define a new class $MHS_S^*(n, \alpha)$ (see [1]).

Definition 1.1: For $0 \le \alpha < 1$, we let $MHS_S^*(n, \alpha)$ denote the class of meromorphic harmonic functions f of the form (1) such that

$$Re\left\{-\frac{2D^{n+1} f(z)}{D^n f(z) - D^n f(-z)}\right\} > \alpha, \quad z \in U\backslash\{0\}. \qquad (3)$$

We let the subclass $\overline{MHS}_S^*(n, \alpha)$ consist of meromorphic harmonic functions $f_n = h_n + \bar{g}_n$ in $MHS_S^*(n, \alpha)$ so that h_n and g_n are of the form

$$h_n(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} a_k\, z^k \qquad (4)$$

and

$$g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k\, z^k, \qquad (5)$$

where $a_k \ge 0, b_k \ge 0$.

In this paper, we have obtained the coefficient conditions for the classes $MHS_S^*(n, \alpha)$ and $\overline{MHS}_S^*(n, \alpha)$.Further a representation theorem, inclusion properties and distortion bound for the class $\overline{MHS}_S^*(n, \alpha)$ are established.

II. Main Results

Theorem 2.1: Let f be of the form (1). If

$$\sum_{k=1}^{\infty} [(|a_{2k}| + |b_{2k}|)(2k)^{n+1} + ((2k - 1 + \alpha)\,|a_{2k-1}| \qquad (6)$$

$$+(2k - 1 - \alpha)\,|b_{2k-1}|)(2k - 1)^n] \le 1 - \alpha,$$

then f is harmonic univalent, sense preserving in $U\backslash\{0\}$ and $f \in MHS_S^*(n, \alpha)$.

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Proof: For $0 < |z_1| \le |z_2| < 1$ we have

$$\begin{aligned}
& |f(z_1) - f(z_2)| \\
\ge\ & |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\
\ge\ & \frac{|z_1 - z_2|}{|z_1||z_2|} \\
& - |z_1 - z_2| \sum_{k=1}^{\infty} (|a_k| + |b_k|) \left| z_1^{k-1} + \cdots + z_2^{k-1} \right| \\
>\ & \frac{|z_1 - z_2|}{|z_1||z_2|} \left[1 - |z_2|^2 \sum_{k=1}^{\infty} k(|a_k| + |b_k|) \right] \\
=\ & \frac{|z_1 - z_2|}{|z_1||z_2|} \left[1 - |z_2|^2 \left(\sum_{k=1}^{\infty} 2k(|a_{2k}| + |b_{2k}|) \right.\right. \\
& \left.\left. + \sum_{k=1}^{\infty} (2k-1)(|a_{2k-1}| + |b_{2k-1}|) \right) \right] \\
>\ & \frac{|z_1 - z_2|}{|z_1||z_2|} \left[1 - \sum_{k=1}^{\infty} (2k)^{n+1} (|a_{2k}| + |b_{2k}|) \right. \\
& - \sum_{k=1}^{\infty} (2k-1)^n [(2k-1+\alpha)|a_{2k-1}|] \\
& \left. - \sum_{k=1}^{\infty} (2k-1)^n [(2k-1-\alpha)|b_{2k-1}|] \right].
\end{aligned}$$

This last expression is non negative by (6) and so f is univalent in $U \backslash \{0\}$. To show that f is sense preserving in $U \backslash \{0\}$, we need to show that $|h'(z)| \ge |g'(z)|$ in $U \backslash \{0\}$. We have

$$\begin{aligned}
|h'(z)| &\ge 1 - \sum_{k=1}^{\infty} k|a_k||z|^{k-1} \\
&= 1 - \sum_{k=1}^{\infty} k|a_k| r^{k-1} > 1 - \sum_{k=1}^{\infty} k|a_k| \\
&\ge 1 - \sum_{k=1}^{\infty} (2k)^{n+1} |a_{2k}| \\
&\quad - \sum_{k=1}^{\infty} (2k-1)^n (2k-1+\alpha) |a_{2k-1}| \\
&\ge \sum_{k=1}^{\infty} (2k)^{n+1} |b_{2k}| \\
&\quad + \sum_{k=1}^{\infty} (2k-1)^n (2k-1-\alpha) |b_{2k-1}| \\
&\ge \sum_{k=1}^{\infty} 2k|b_{2k}| + \sum_{k=1}^{\infty} (2k-1)|b_{2k-1}| \\
&> \sum_{k=1}^{\infty} k|b_k| r^{k-1} = \sum_{k=1}^{\infty} k|b_k|\,|z|^{k-1} \ge |g'(z)|.
\end{aligned}$$

Now, we will show that $f \in MHS_S^*(n, \alpha)$. According to (2) and (3), for $0 \le \alpha < 1$, we have

$$\begin{aligned}
& Re\left\{ - \frac{2D^{n+1} f(z)}{D^n f(z) - D^n f(-z)} \right\} \\
=\ & Re\left\{ - \frac{2D^{n+1} h(z) - 2(-1)^n \overline{D^{n+1} g(z)}}{T^n(z)} \right\} \ge \alpha,
\end{aligned}$$

where

$$\begin{aligned}
T^n(z) &= D^n h(z) + (-1)^n \overline{D^n g(z)} \\
&\quad - D^n h(-z) - (-1)^n \overline{D^n g(-z)}.
\end{aligned}$$

Using the fact that $Re\{w\} \ge \alpha$ if and only if $|1 - \alpha + w| \ge |1 + \alpha - w|$, it suffices to show that

$$\begin{aligned}
& \left| 1 - \alpha - \frac{2D^{n+1} f(z)}{D^n f(z) - D^n f(-z)} \right| \\
\ge & \left| 1 + \alpha + \frac{2D^{n+1} f(z)}{D^n f(z) - D^n f(-z)} \right|
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
& \left| 2D^{n+1} f(z) - (1-\alpha)(D^n f(z) - D^n f(-z)) \right| \\
& - \left| 2D^{n+1} f(z) + (1+\alpha)(D^n f(z) - D^n f(-z)) \right| \ge 0. \qquad (7)
\end{aligned}$$

Substituting for $D^n f(z)$ and $D^{n+1} f(z)$ in (7) yields

$$\begin{aligned}
& \left| \frac{2(-1)^n}{z} - 2\sum_{k=1}^{\infty} k^{n+1} a_k z^k + 2(-1)^n \sum_{k=1}^{\infty} k^{n+1} \bar{b}_k \bar{z}^k \right. \\
& + (1-\alpha) \left[\frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n a_k z^k \right. \\
& + (-1)^n \sum_{k=1}^{\infty} k^n \bar{b}_k \bar{z}^k + \frac{(-1)^n}{z} - \sum_{k=1}^{\infty} (-1)^k k^n a_k z^k \\
& \left.\left. - (-1)^n \sum_{k=1}^{\infty} (-1)^k k^n \bar{b}_k \bar{z}^k \right] \right| \\
& - \left| \frac{2(-1)^n}{z} - 2\sum_{k=1}^{\infty} k^{n+1} a_k z^k \right. \\
& + 2(-1)^n \sum_{k=1}^{\infty} k^{n+1} \bar{b}_k \bar{z}^k - (1+\alpha) \left[\frac{(-1)^n}{z} \right. \\
& + \sum_{k=1}^{\infty} k^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} k^n \bar{b}_k \bar{z}^k + \frac{(-1)^n}{z} \\
& \left.\left. - \sum_{k=1}^{\infty} (-1)^k k^n a_k z^k - (-1)^n \sum_{k=1}^{\infty} (-1)^k k^n \bar{b}_k \bar{z}^k \right] \right| \\
=\ & \left| \frac{2(2-\alpha)(-1)^n}{z} \right. \\
& - \sum_{k=1}^{\infty} (2k - (1-\alpha) + (-1)^k (1-\alpha)) k^n a_k z^k) \\
& \left. + (-1)^n \sum_{k=1}^{\infty} (2k + (1-\alpha) - (-1)^k (1-\alpha) k^n \bar{b}_k \bar{z}^k \right| \\
& - \left| \frac{2\alpha(-1)^n}{z} + \sum_{k=1}^{\infty} (2k + (1+\alpha) - (-1)^k (1+\alpha)) k^n a_k z^k \right. \\
& \left. - (-1)^n \sum_{k=1}^{\infty} (2k - (1-\alpha) + (-1)^k (1+\alpha)) k^n \bar{b}_k \bar{z}^k \right| \\
=\ & \left| \frac{2(2-\alpha)(-1)^n}{z} \right.
\end{aligned}$$

$$
\begin{aligned}
& -2\sum_{k=1}^{\infty}(2k-2+\alpha)(2k-1)^n a_{2k-1}\, z^{2k-1} \\
& -2\sum_{k=1}^{\infty}(2k)^{n+1} a_{2k}\, z^{2k} + 2(-1)^n\sum_{k=1}^{\infty}(2k)^{n+1}\bar{b}_{2k}\bar{z}^{2k} \\
& +2(-1)^n\sum_{k=1}^{\infty}(2k-\alpha)(2k-1)^n\bar{b}_{2k-1}\bar{z}^{2k-1}\Bigg| \\
& -\Bigg|\frac{2\alpha(-1)^n}{z} + 2\sum_{k=1}^{\infty}(2k)^{n+1}a_{2k}\, z^{2k} \\
& +2\sum_{k=1}^{\infty}(2k+\alpha)(2k-1)^n a_{2k-1}\, z^{2k-1} \\
& -2(-1)^n\sum_{k=1}^{\infty}(2k)^{n+1}\bar{b}_{2k}\bar{z}^{2k} \\
& -2(-1)^n\sum_{k=1}^{\infty}(2k-2+\alpha)(2k-1)^n\bar{b}_{2k-1}\bar{z}^{2k-1}\Bigg| \\
\geq\ & \frac{2(2-\alpha)(-1)^n}{z} \\
& -2\sum_{k=1}^{\infty}(2k-2+\alpha)(2k-1)^n|a_{2k-1}|\,|z|^{2k-1} \\
& -2\sum_{k=1}^{\infty}(2k)^{n+1}|a_{2k}||\,z|^{2k} - 2\sum_{k=1}^{\infty}(2k)^{n+1}|b_{2k}||z|^{2k} \\
& -2\sum_{k=1}^{\infty}(2k-\alpha)(2k-1)^n|b_{2k-1}||z|^{2k-1} \\
& -\frac{2\alpha(-1)^n}{z} - 2\sum_{k=1}^{\infty}(2k+\alpha)(2k-1)^n|a_{2k-1}||\,z|^{2k-1} \\
& -2\sum_{k=1}^{\infty}(2k)^{n+1}|a_{2k}||z|^{2k} - 2\sum_{k=1}^{\infty}(2k)^{n+1}|b_{2k}||z|^{2k} \\
& -2\sum_{k=1}^{\infty}(2k-2+\alpha)(2k-1)^n|b_{2k-1}||z|^{2k-1} \\
=\ & \frac{4(1-\alpha)}{z}\left[1-\sum_{k=1}^{\infty}\left\{\frac{(2k)^{n+1}}{1-\alpha}(|a_{2k}|+|b_{2k}|)\right.\right. \\
& +\frac{(2k-1)^n}{1-\alpha}[(2k-1+\alpha)|a_{2k-1}| \\
& \left.\left. +(2k-1-\alpha)|b_{2k-1}|]\right\}\right].
\end{aligned}
$$

This last expression is non-negative by (7), and so the proof is complete. ■

Theorem 2.2: Let $f_n = h_n + \overline{g_n}$ where h_n and g_n are of the form (4) and (5). Then $f_n \in \overline{MHS}^*_S(n,\alpha)$, if and only if

$$
\begin{aligned}
& \sum_{k=1}^{\infty}[(a_{2k}+b_{2k})(2k)^{n+1} + ((2k-1+\alpha)a_{2k-1} \qquad (8) \\
& +(2k-1-\alpha)b_{2k-1})(2k-1)^n] \leq 1-\alpha.
\end{aligned}
$$

Proof: Since $\overline{MHS}^*_S(n,\alpha) \subset MHS^*_S(n,\alpha)$, we only need to prove the (only if) part of the theorem. To this end, for functions $f_n = h_n + \overline{g}_n$, we notice that condition

$$Re\left\{-\frac{2D^{n+1}f(z)}{D^n f(z) - D^n f(-z)}\right\} \geq \alpha, \quad z \in U\backslash\{0\},$$

is equivalent to

$$
\begin{aligned}
& Re\left\{\frac{\frac{2(1-\alpha)}{z} - \sum_{k=1}^{\infty}(2k+\alpha-(-1)^k\alpha)k^n a_k z^k}{\phi(z)}\right. \\
& \left. +\frac{(-1)^n\sum_{k=1}^{\infty}(2k-\alpha+(-1)^k\alpha)k^n b_k \bar{z}^k}{\phi(z)}\right\} \geq 0
\end{aligned}
$$

which implies

$$
\begin{aligned}
& Re\left\{\frac{\frac{2(1-\alpha)}{z} - 2\sum_{k=1}^{\infty}(2k)^{n+1}a_{2k}z^{2k}}{\phi(z)}\right. \\
& -\frac{2\sum_{k=1}^{\infty}(2k-1+\alpha)(2k-1)^n a_{2k-1}z^{2k-1}}{\phi(z)} \\
& -\frac{2(-1)^n\sum_{k=1}^{\infty}(2k-1-\alpha)(2k-1)^n b_{2k-1}\bar{z}^{2k-1}}{\phi(z)} \\
& \left. -\frac{2(-1)^n\sum_{k=1}^{\infty}(2k)^{n+1}b_{2k}z^{2k}}{\phi(z)}\right\} \geq 0, \qquad (9)
\end{aligned}
$$

where

$$
\begin{aligned}
\phi(z) &= \frac{2}{z} + 2\sum_{k=1}^{\infty}(2k-1)^n a_{2k-1}z^{2k-1} \\
&\quad +2\sum_{k=1}^{\infty}(2k-1)^n b_{2k-1}\bar{z}^{2k-1}.
\end{aligned}
$$

The condition (9) must hold for all z in $U\backslash\{0\}$. By choosing $0 < z = r < 1$, from the left hand (9), we have

$$
\begin{aligned}
& \frac{1-\alpha-\sum_{k=1}^{\infty}(2k-1+\alpha)(2k-1)^n a_{2k-1}r^{2k}}{r\phi(r)/2} \\
& -\frac{\sum_{k=1}^{\infty}(2k)^{n+1}a_{2k}r^{2k+1} + (-1)^n\sum_{k=1}^{\infty}(2k)^{n+1}b_{2k}r^{2k+1}}{r\phi(r)/2} \\
& -\frac{(-1)^n\sum_{k=1}^{\infty}(2k-1-\alpha)(2k-1)^n b_{2k-1}r^{2k}}{r\phi(r)/2}. \qquad (10)
\end{aligned}
$$

If the condition (8) does not hold, then the number in (10) is negative for r sufficiently close to 1. Hence there exist

$z_0 = r_0$ in $(0,1)$, for which the eqnarray in (10) is negative. This contradicts the required condition for and so the proof is complete. It is easily seen that $f_n(z) \in \overline{MHS}^*_S(n,\alpha)$. Thus we complete the of the Theorem 2.2. ∎

Theorem 2.3: If $f_n = h_n + \bar{g}_n \in \overline{MHS}^*_S(n,\alpha)$ for $0 < |z| = r < 1$, then

$$\frac{1}{r} - \frac{1-\alpha}{2^{n+1}} r \le |f_n(z)| \le \frac{1}{r} + \frac{1-\alpha}{2^{n+1}} r.$$

Proof: Let $f_n = h_n + \bar{g}_n \in \overline{MHS}^*_S(n,\alpha)$. Taking the absolute value of f we obtain

$$\begin{aligned} |f_n(z)| &= \left| \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} \overline{b_k z^k} \right| \\ &\le \frac{1}{r} + \sum_{k=1}^{\infty} (a_k + b_k) r^k \\ &\le \frac{1}{r} + \sum_{k=1}^{\infty} (a_k + b_k) r \\ &\le \frac{1}{r} + \frac{1-\alpha}{2^{n+1}} \sum_{k=1}^{\infty} \frac{2^{n+1}}{1-\alpha} (|a_k| + |b_k|) r \\ &\le \frac{1}{r} + \frac{1-\alpha}{2^{n+1}} r \end{aligned}$$

and

$$\begin{aligned} |f_n(z)| &= \left| \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} \overline{b_k z^k} \right| \\ &\ge \frac{1}{r} - \sum_{k=1}^{\infty} (a_k + b_k) r^k \\ &\ge \frac{1}{r} - \sum_{k=1}^{\infty} (a_k + b_k) r \\ &\ge \frac{1}{r} - \frac{1-\alpha}{2^{n+1}} \sum_{k=1}^{\infty} \frac{2^{n+1}}{1-\alpha} (|a_k| + |b_k|) r \\ &\ge \frac{1}{r} - \frac{1-\alpha}{2^{n+1}} r. \end{aligned}$$

∎

Corollary 2.4: Let $A = \left\{ w : |w| < \frac{2^{n+1} - 1 + \alpha}{2^{n+1}} \right\}$. If $f_n = h_n + \bar{g}_n \in \overline{MHS}^*_S(n,\alpha)$, then

$$f_n(U) \subset A^t.$$

Theorem 2.5: $f_n = h_n + \bar{g}_n \in \overline{MHS}^*_S(n,\alpha)$ if and only if f_n can be expressed as

$$f_n(z) = \sum_{k=0}^{\infty} (x_k h_{n_k} + y_k g_{n_k}), \tag{11}$$

where for $k = 1, 2, \cdots$

$$\begin{aligned} h_{n_0}(z) &= g_{n_0}(z) = \frac{1}{z} \\ h_{n_{2k-1}}(z) &= \frac{1}{z} + \frac{1-\alpha}{(2k-1)^n (2k-1+\alpha)} z^{2k-1} \\ h_{n_{2k}}(z) &= \frac{1}{z} + \frac{1-\alpha}{(2k)^{n+1}} z^{2k} \\ g_{n_{2k-1}}(z) &= \frac{1}{z} + \frac{1-\alpha}{(2k-1)^n (2k-1-\alpha)} \bar{z}^{2k-1} \\ g_{n_{2k}}(z) &= \frac{1}{z} + \frac{1-\alpha}{(2k)^{n+1}} \bar{z}^{2k} \end{aligned}$$

and

$$\sum_{k=0}^{\infty} (x_k + y_k) = 1, \;\; x_k \ge 0 \;\; and \; y_k \ge 0.$$

In particular, the extreme point of $\overline{MHS}^*_S(n,\alpha)$ are $\{h_{n_k}\}$ and $\{g_{n_k}\}$.

Proof: For functions $f_n = h_n + \bar{g}_n$, where h_n and g_n of the form (4) and (5), we have

$$\begin{aligned} f_n(z) &= \sum_{k=0}^{\infty} (x_k h_{n_k} + y_k g_{n_k}) \\ &= x_0 h_{n_0} + y_0 g_{n_0} + \sum_{k=1}^{\infty} (x_k + y_k) \frac{1}{z} \\ &\quad + \sum_{k=1}^{\infty} x_{2k-1} \frac{1-\alpha}{(2k-1)^n (2k-1+\alpha)} z^{2k-1} \\ &\quad + \sum_{k=1}^{\infty} x_{2k} \frac{1-\alpha}{(2k)^{n+1}} z^{2k} \\ &\quad + \sum_{k=1}^{\infty} y_{2k-1} \frac{1-\alpha}{(2k-1)^n (2k-1-\alpha)} \bar{z}^{2k-1} \\ &\quad + \sum_{k=1}^{\infty} y_{2k} \frac{1-\alpha}{(2k)^{n+1}} \bar{z}^{2k} \\ &= \sum_{k=0}^{\infty} (x_k + y_k) \frac{1}{z} \\ &\quad + \sum_{k=1}^{\infty} \frac{1-\alpha}{(2k-1)^n (2k-1+\alpha)} x_{2k-1} z^{2k-1} \\ &\quad + \sum_{k=1}^{\infty} \frac{1-\alpha}{(2k)^{n+1}} x_{2k} z^{2k} \\ &\quad + \sum_{k=1}^{\infty} \frac{1-\alpha}{(2k-1)^n (2k-1-\alpha)} y_{2k-1} \bar{z}^{2k-1} \\ &\quad + \sum_{k=1}^{\infty} \frac{1-\alpha}{(2k)^{n+1}} y_{2k} \bar{z}^{2k}. \end{aligned}$$

Then,

$$\begin{aligned} &\sum_{k=1}^{\infty} (2k-1+\alpha)(2k-1)^n \\ &\times \left\{ \frac{1-\alpha}{(2k-1)^n (2k-1+\alpha)} x_{2k-1} \right\} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{\infty}(2k-1-\alpha)(2k-1)^n \\
&\times\left\{\frac{1-\alpha}{(2k-1)^n(2k-1-\alpha)}y_{2k-1}\right\} \\
&+\sum_{k=1}^{\infty}(2k)^{n+1}\left\{\frac{1-\alpha}{(2k)^{n+1}}(x_{2k}+y_{2k})\right\} \\
= &\ (1-\alpha)\sum_{k=1}^{\infty}(x_k+y_k) \\
= &\ (1-\alpha)[1-(x_0+y_0)] \le 1-\alpha.
\end{aligned}
$$

So $f_n \in \overline{MHS}^*_S(n,\alpha)$.

Conversely, suppose that $f_n \in \overline{MHS}^*_S(n,\alpha)$. Let, for $k = 1, 2, \ldots$

$$
\begin{aligned}
x_{2k} &= \frac{1-\alpha}{(2k)^{n+1}}a_{2k},\, y_{2k} = \frac{1-\alpha}{(2k)^{n+1}}b_{2k} \\
x_{2k-1} &= \frac{1-\alpha}{(2k-1)^n(2k-1+\alpha)}a_{2k-1} \\
y_{2k-1} &= \frac{1-\alpha}{(2k-1)^n(2k-1-\alpha)}b_{2k-1}.
\end{aligned}
$$

Then note that by the $\overline{MHS}^*_S(n,\alpha)$, $0 \le x_k \le 1$ and $0 \le y_k \le 1$ $(k = 1, 2, \ldots)$. We define $0 \le x_0 \le 1$ and

$$y_0 = 1 - x_0 - \sum_{k=1}^{\infty}(x_k + y_k).$$

Consequently, we obtain

$$f(z) = \sum_{k=0}^{\infty}(x_k h_{n_k} + y_k g_{n_k})$$

as required. ■

Theorem 2.6: If $f \in MHS^*_S(n,\alpha)$, then the diameter D_f of $\mathbf{C}\backslash f(U)$ satisfies

$$D_f \ge 2|1+b_1|.$$

Proof: Let $D_f(R)$ be diameter of $f(|z| = R), 0 < R < 1$, and let $D^*_f(R) = \max_{|z|=R}|f(z) - f(-z)|$. Then $D_f(R) \to D_f$ as $R \to 1$ and $D_f(R) \ge D^*_f(R)$. Since

$$
\begin{aligned}
[D^*_f(R)]^2 &\ge \frac{1}{2\pi}\int_0^{2\pi}|f(Re^{i\theta}) - f(-Re^{i\theta})|^2 d\theta \\
&= 4\left[\frac{1}{R^2} + b_1 + \bar{b}_1\right. \\
&\quad \left.+\sum_{k=1}^{\infty}(|a_{2k-1}|^2 + |b_{2k-1}|^2)R^{2(2n-1)}\right] \\
&\ge 4\left[1 + 2Reb_1\right. \\
&\quad \left.+\sum_{k=1}^{\infty}(|a_{2k-1}|^2 + |b_{2k-1}|^2)\right]
\end{aligned}
$$

we conclude that $D_f \ge 2\sqrt{|1+b_1|^2}$. ■

Note that if f and F are

$$f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + \sum_{k=1}^{\infty}a_k\, z^k + \overline{\sum_{k=1}^{\infty}b_k\, z^k}$$

and

$$F(z) = H(z) + \overline{G(z)} = \frac{1}{z} + \sum_{k=1}^{\infty}A_k\, z^k + \overline{\sum_{k=1}^{\infty}B_k\, z^k},$$

then the convolution (or Hadamard product) of f and F if defined to be the function

$$(f * F)(z) = \frac{1}{z} + \sum_{k=1}^{\infty}a_k A_k\, z^k + \overline{\sum_{k=1}^{\infty}b_k\, B_k z^k}. \tag{12}$$

Theorem 2.7: For $0 \le \beta \le \alpha < 1$, let $f_n \in \overline{MHS}^*_S(n,\alpha)$ and $F_n \in \overline{MHS}^*_S(n,\beta)$.Then the convolution function $f_n * F_n \in \overline{MHS}^*_S(n,\alpha) \subset \overline{MHS}^*_S(n,\beta)$.

Proof: For f_n and F_n as Theorem 2.7. Then the convolution $f_n * F_n$ is given by (12). We wish to show that the coefficients of $f_n * F_n$ satisfy the required condition given in Theorem 2.2. For $F_n \in \overline{MHS}^*_S(n,\beta)$, we note that $|A_k| \le 1$ and $|B_k| \le 1$. Since $0 \le \beta \le \alpha < 1$ and $f_n \in \overline{MHS}^*_S(n,\alpha)$ for $f_n * F_n$, we obtain

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\left[(a_{2k}A_{2k} + b_{2k}B_{2k})\frac{(2k)^{n+1}}{1-\beta}\right. \\
&\quad +((2k-1+\alpha)a_{2k-1}A_{2k-1} \\
&\quad \left.+(2k-1-\alpha)b_{2k-1}B_{2k-1})\frac{(2k-1)^n}{1-\beta}\right] \\
\le &\sum_{k=1}^{\infty}\left[(a_{2k} + b_{2k})\frac{(2k)^{n+1}}{1-\beta}\right. \\
&\quad +((2k-1+\alpha)a_{2k-1} \\
&\quad \left.+(2k-1-\alpha)b_{2k-1})\frac{(2k-1)^n}{1-\beta}\right] \\
\le &\sum_{k=1}^{\infty}\left[(a_{2k} + b_{2k})\frac{(2k)^{n+1}}{1-\alpha}\right. \\
&\quad +((2k-1+\alpha)a_{2k-1} \\
&\quad \left.+(2k-1-\alpha)b_{2k-1})\frac{(2k-1)^n}{1-\alpha}\right] \\
\le &\ 1.
\end{aligned}
$$

Therefore $f_n * F_n \in \overline{MHS}^*_S(n,\alpha) \subset \overline{MHS}^*_S(n,\beta)$. ■

Theorem 2.8: The class $\overline{MHS}^*_S(n,\alpha)$ is closed under convex combination.

Proof: Suppose that $f_{n_i}(z) \in \overline{MHS}^*_S(n,\alpha)$ for $i = 1, 2, 3, \cdots$, where f_{n_i} is given by

$$f_{n_i}(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty}a_{i_k}z^k + (-1)^n\overline{\sum_{k=1}^{\infty}b_{i_k}z^k}.$$

Then by Theorem 2.2.

$$\sum_{k=1}^{\infty}[(a_{i_{2k}} + b_{i_{2k}})(2k)^{n+1} + ((2k-1+\alpha)a_{i_{2k-1}} \tag{13}$$

$$+(2k-1-\alpha)b_{i_{2k-1}})(2k-1)^n] \leq 1-\alpha.$$

For

$$\sum_{i=1}^{\infty} t_i = 1,\ 0 \leq t_i \leq 1,$$

the convex combinations of f_{n_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{i_k}\right) z^{-n} + (-1)^n \overline{\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i\, b_{i_k}\right) z^{-n}}.$$

Then by (13),

$$\sum_{k=1}^{\infty}\left[\frac{(2k)^{n+1}}{1-\alpha}\sum_{i=1}^{\infty} t_i\left(a_{i_{2k}}+b_{i_{2k}}\right) + \frac{(2k-1)^n}{1-\alpha}\left(\frac{(2k-1+\alpha)}{1-\alpha}\sum_{i=1}^{\infty} t_i a_{i_{2k-1}} + \frac{(2k-1-\alpha)}{1-\alpha}\sum_{i=1}^{\infty} t_i b_{i_{2k-1}}\right)\right] \leq \sum_{i=1}^{\infty} t_i = 1.$$

Thus

$$\sum_{i=1}^{\infty} t_i\ f_{n_i}(z) \in \overline{MHS}_S^*(n,\alpha).$$

■

Theorem 2.9: If $f_n \in \overline{MHS}_S^*(n,\alpha)$, then

$$\sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) \leq 1 + 2Re\{b_1\}.$$

Equality occurs if and only if $\mathbf{C}\backslash f(U)$ has area zero.

Proof: The area of the omitted set is

$$\lim_{R\to 1} \lim \frac{1}{2i}\int_{0<|z|=R<1} \bar{f} df$$

$$= \lim_{R\to 1}\left[\frac{1}{2i}\int_{0<|z|=R<1} \bar{h}h' dz + \frac{1}{2i}\int_{0<|z|=R<1} g\overline{g'} d\bar{z} + \frac{1}{2i}\int_{0<|z|=R<1} gh' dz + \frac{1}{2i}\int_{0<|z|=R<1} \bar{h}\overline{g'} d\bar{z}\right]$$

$$= \pi\left[\sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2)R^{2k} - \frac{1}{R^2} - 2Reb_1\right].$$

For $0 < r < 1$ the curve $\Gamma_r = f(C_r)$ is a simple closed curve oriented clockwise. Hence, for $R \to 1$ we obtain

$$\sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) - 1 - 2Re\{b_1\} \leq 0$$

and the result follows. ■

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