Some Collineations Preserving Cross-Ratio in some Moufang-Klingenberg Planes

Süleyman Ciftci, Atilla Akpinar and Basri Celik

***Abstract*—In this paper we are interested in Moufang-Klingenberg planes $\mathbf{M}(\mathcal{A})$ defined over a local alternative ring $\mathcal{A}$ of dual numbers. We show that some collineations of $\mathbf{M}(\mathcal{A})$ preserve cross-ratio.**

***Keywords*—Moufang-Klingenberg planes, local alternative ring, projective collineation, cross-ratio.**

I. Introduction

The number of collineations of any projective plane is huge. For example; the Fano plane has 168 collineations, the non-Desarguesian projective Veblen-Wedderburn plane of order 9 (which is denoted by $\pi_N(9)$) has 311,040 collineations [14, p. 366]. It is easy to see that the composite of any two collineations is a collineation, as the invers of any collineation. Function composition is always associative; thus the collineations of any projective or affine plane form a group. For more detailed information about these groups, the reader is referred to the books of [11], [14].

In the Euclidean plane, Desargues established the fundemantal fact that cross-ratio (a concept originally introduced by Pappus of Alexandria c.300 B.C) is invariant under projection [3, p. 133]. For this reason, cross-ratio is one of the most important concepts of projective geometry.

In this paper we deal with the class (which we will denote by $\mathbf{M}(\mathcal{A})$) of Moufang-Klingenberg (MK) planes coordinatized by a local alternative ring

$$\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$$

(an alternative field $\mathbf{A}$, $\varepsilon \notin \mathbf{A}$ and $\varepsilon^2 = 0$) introduced by Blunck in [7]. We will show that some collineations of $\mathbf{M}(\mathcal{A})$ from [8] preserve cross-ratio. For more information about some well-known properties of cross-ratio in the case of Moufang planes or MK-planes $\mathbf{M}(\mathcal{A})$, respectively, it can be seen the papers of [10], [4], [9] or [7], [1].

Section 2 includes some basic definitions and results from the literature.

In Section 3 we will give some collineations of $\mathbf{M}(\mathcal{A})$ from [8] and we show that the collineations preserve cross-ratio, the main result of the paper.

II. Preliminaries

Let $\mathbf{M} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$ (points, lines, incidence) and an equivalence relation '$\sim$' (neighbour relation) on $\mathbf{P}$ and on $\mathbf{L}$, respectively. Then $\mathbf{M}$ is called a *projective Klingenberg plane* (PK-plane), if it satisfies the following axioms:

(PK1) If P, Q are non-neighbour points, then there is a unique line PQ through P and Q.

(PK2) If g, h are non-neighbour lines, then there is a unique point $g \cap h$ on both g and h.

(PK3) There is a projective plane $\mathbf{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$ and an incidence structure epimorphism $\Psi : \mathbf{M} \to \mathbf{M}^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q,\ \Psi(g) = \Psi(h) \iff g \sim h$$

hold for all $P, Q \in \mathbf{P}$, $g, h \in \mathbf{L}$.

A point $P \in \mathbf{P}$ is called *near* a line $g \in \mathbf{L}$ iff there exists a line $h \sim g$ such that $P \in h$.

Let $h, k \in \mathbf{L}$, $C \in \mathbf{P}$, C is not near to h, k. Then the well-defined bijection

$$\sigma := \sigma_C(k, h) : \begin{cases} h \to k \\ X \to XC \cap k \end{cases}$$

mapping h to k is called a *perspectivity* from h to k with center C. A product of a finite number of perspectivities is called a *projectivity*.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a *collineation* of $\mathbf{M}$.

A *Moufang-Klingenberg plane* (MK-plane) is a PK-plane $\mathbf{M}$ that generalizes a Moufang plane, and for which $\mathbf{M}^*$ is a Moufang plane (for the exact definition see [2]).

An *alternative ring (field)* $\mathbf{R}$ is a not necessarily associative ring (field) that satisfies the alternative laws

$$a(ab) = a^2 b, (ba)a = ba^2, \forall a, b \in \mathbf{R}.$$

An alternative ring $\mathbf{R}$ with identity element 1 is called *local* if the set $\mathbf{I}$ of its non-unit elements is an ideal.

We are now ready to give consecutively two important lemmas related to alternative rings.

Lemma 2.1: The subring generated by any two elements of an alternative ring is associative (cf. [13, Theorem 3.1]).

Lemma 2.2: The identities

$$\begin{aligned} x(y(xz)) &= (xyx)z \\ ((yx)z)x &= y(xzx) \\ (xy)(zx) &= x(yz)x \end{aligned}$$

which are known as Moufang identities are satisfied in every alternative ring (cf. [12, p. 160]).

Süleyman Ciftci, Atilla Akpinar and Basri Celik are with the Uludag University, Department of Mathematics, Faculty of Arts and Science, Bursa-TURKEY, email: sciftci@uludag.edu.tr, aakpinar@uludag.edu.tr, basri@uludag.edu.tr

We summarize some basic concepts about the coordinatization of MK-planes from [2].

Let $\mathbf{R}$ be a local alternative ring. Then $\mathbf{MR} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is the incidence structure with neighbour relation defined as follows:

$$\begin{aligned}
\mathbf{P} &= \{(x, y, 1) | x, y \in \mathbf{R}\} \cup \{(1, y, z) | y \in \mathbf{R} z \in \mathbf{I}\} \\
&\quad \cup \{(w, 1, z) | w, z \in \mathbf{I}\}, \\
\mathbf{L} &= \{[m, 1, p] | m, p \in \mathbf{R}\} \cup \{[1, n, p] | p \in \mathbf{R} n \in \mathbf{I}\} \\
&\quad \cup \{[q, n, 1] | q, n \in \mathbf{I}\}, \\
[m, 1, p] &= \{(x, xm + p, 1) \,| x \in \mathbf{R}\} \\
&\quad \cup \{(1, zp + m, z) \,| z \in \mathbf{I}\}, \\
[1, n, p] &= \{(yn + p, y, 1) \,| y \in \mathbf{R}\} \\
&\quad \cup \{(zp + n, 1, z) \,| z \in \mathbf{I}\}, \\
[q, n, 1] &= \{(1, y, yn + q) \,| y \in \mathbf{R}\} \\
&\quad \cup \{(w, 1, wq + n) \,| w \in \mathbf{I}\}, \\
P &= (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = Q \\
&\quad \Leftrightarrow x_i - y_i \in \mathbf{I} \;\; (i = 1, 2, 3)), \forall P, Q \in \mathbf{P}, \\
g &= [x_1, x_2, x_3] \sim [y_1, y_2, y_3] = h \\
&\quad \Leftrightarrow x_i - y_i \in \mathbf{I} \;\; (i = 1, 2, 3)), \forall g, h \in \mathbf{L}.
\end{aligned}$$

Now it is time to give the following theorem from [2].

Theorem 2.1: $\mathbf{M}(\mathbf{R})$ is an MK-plane, and each MK-plane is isomorphic to some $\mathbf{M}(\mathbf{R})$.

Let $\mathbf{A}$ be an alternative field and $\varepsilon \notin \mathbf{A}$. Consider

$$\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$$

with componentwise addition and multiplication as follows:

$$(a_1 + a_2\varepsilon)(b_1 + b_2\varepsilon) = a_1 b_1 + (a_1 b_2 + a_2 b_1)\varepsilon,$$

where $a_i, b_i \in \mathbf{A}$ for $i = 1, 2$. Then $\mathcal{A}$ is a local alternative ring with ideal $\mathbf{I} = \mathbf{A}\varepsilon$ of non-units. The set of formal inverses of the non-units of $\mathcal{A}$ is denoted as $\mathbf{I}^{-1}$. Calculations with the elements of $\mathbf{I}^{-1}$ are defined as follows [6]:

$$\begin{aligned}
(a\varepsilon)^{-1} + t &:= (a\varepsilon)^{-1} := t + (a\varepsilon)^{-1} \\
q\,(a\varepsilon)^{-1} &:= \left(aq^{-1}\varepsilon\right)^{-1} \\
(a\varepsilon)^{-1} q &:= \left(q^{-1}a\varepsilon\right)^{-1} \\
\left((a\varepsilon)^{-1}\right)^{-1} &:= a\varepsilon,
\end{aligned}$$

where $(a\varepsilon)^{-1} \in \mathbf{I}^{-1}, t \in \mathcal{A}$, $q \in \mathcal{A} \setminus \mathbf{I}$. (Other terms are not defined.). For more information about $\mathcal{A}$ and its relation to MK-planes, the reader is referred to the papers of Blunck [6], [7]. In [7], the centre $\mathbf{Z}(\mathcal{A})$ is defined to be the (commutative, associative) subring of $\mathcal{A}$ which is commuting and associating with all elements of $\mathcal{A}$. It is $\mathbf{Z}(\mathcal{A}) := \mathbf{Z}(\varepsilon) = \mathbf{Z} + \mathbf{Z}\varepsilon$, where $\mathbf{Z} = \{z \in \mathbf{A} | za = az, \; \forall a \in \mathbf{A}\}$ is the centre of $\mathbf{A}$. If $\mathbf{A}$ is not associative, then $\mathbf{A}$ is a Cayley division algebra over its centre $\mathbf{Z}$.

Throughout this paper we assume $char\mathbf{A} \neq \mathbf{2}$ and we restrict ourselves to the MK-planes $\mathbf{M}(\mathcal{A})$.

Blunck [7] gives the following algebraic definition of the cross-ratio for the points on the line $g := [1, 0, 0]$ in $\mathbf{M}(\mathcal{A})$.

$$\begin{aligned}
&(A, B; C, D) := (a, b; c, d) \\
&=< \left((a - d)^{-1} (b - d)\right) \left((b - c)^{-1} (a - c)\right) > \\
&(Z, B; C, D) := \left(z^{-1}, b; c, d\right) \\
&=< \left((1 - dz)^{-1} (b - d)\right) \left((b - c)^{-1} (1 - cz)\right) > \\
&(A, Z; C, D) := \left(a, z^{-1}; c, d\right) \\
&=< \left((a - d)^{-1} (1 - dz)\right) \left((1 - cz)^{-1} (a - c)\right) > \\
&(A, B; Z, D) := \left(a, b; z^{-1}, d\right) \\
&=< \left((a - d)^{-1} (b - d)\right) \left((1 - zb)^{-1} (1 - za)\right) > \\
&(A, B; C, Z) := \left(a, b; c, z^{-1}\right) \\
&=< \left((1 - za)^{-1} (1 - zb)\right) \left((b - c)^{-1} (a - c)\right) >,
\end{aligned}$$

where $A = (0, a, 1)$, $B = (0, b, 1)$, $C = (0, c, 1)$, $D = (0, d, 1)$, $Z = (0, 1, z)$ are pairwise non-neighbour points of g and $< x >= \left\{y^{-1}xy | \;\; y \in \mathcal{A}\right\}$.

In [6, Theorem 2], it is shown that the transformations

$$\begin{aligned}
t_u(x) &= x + u; \; u \in \mathcal{A} \\
r_u(x) &= xu; \; u \in \mathcal{A} \setminus \mathbf{I} \\
i(x) &= x^{-1} \\
l_u(x) &= ux = \left(ir_u^{-1}i\right)(x); \; u \in \mathcal{A} \setminus \mathbf{I}
\end{aligned}$$

which are defined on the line g preserve cross-ratios. In [5, Corollary (iii)], it is also shown that the group generated by these transformations, which is denoted by Λ, equals to the group of projectivities of a line in $\mathbf{M}(\mathcal{A})$. The elements preserving cross-ratio of the group Λ defined on g will act a very important role in the proof of Theorem 3.1.

We give the following result from [1, Theorem 8]. This result states a simple way for calculation of the cross-ratio of the points on any line in $\mathbf{M}(\mathcal{A})$.

Theorem 2.2: Let $\{O, U, V, E\}$ be the basis of $\mathbf{M}(\mathcal{A})$ where $O = (0, 0, 1), U = (1, 0, 0), V = (0, 1, 0), E = (1, 1, 1)$ (see [2, Section 4]). Then, according to types of lines, the cross-ratio of the points on the line l can be calculated as follows:

If A, B, C, D and Z are the pairwise non-neighbour points

(a) of the line $l = [m, 1, k]$, where $A = (a, am + k, 1)$, $B = (b, bm + k, 1)$, $C = (c, cm + k, 1)$, $D = (d, dm + k, 1)$ are not near to the line $UV = [0, 0, 1]$ and $Z = (1, m + zp, z)$ is near to UV,
(b) of the line $l = [1, n, p]$, where $A = (an + p, a, 1)$, $B = (bn + p, b, 1)$, $C = (cn + p, c, 1)$, $D = (dn + p, d, 1)$ are not neighbour to V and $Z = (n + zp, 1, z) \sim V$,
(c) of the line $l = [q, n, 1]$, where $A = (1, a, q + an)$, $B = (1, b, q + bn)$, $C = (1, c, q + cn)$, $D = (1, d, q + dn)$ are not neighbour to V and $Z = (z, 1, zq + n) \sim V$,

then

$$\begin{aligned}(A,B;C,D) &= (a,b;c,d)\\ (Z,B;C,D) &= \left(z^{-1},b;c,d\right)\\ (A,Z;C,D) &= \left(a,z^{-1};c,d\right)\\ (A,B;Z,D) &= \left(a,b;z^{-1},d\right)\\ (A,B;C,Z) &= \left(a,b;c,z^{-1}\right).\end{aligned}$$

We can give an important theorem, from [1, Theorem 9], about cross-ratio.

Theorem 2.3: In $\mathbf{M}(\mathcal{A})$, perspectivities preserve cross-ratios.

In the next section, we deal with some collineations preserving cross-ratio in $\mathbf{M}(\mathcal{A})$.

III. Some Collineations Preserving Cross-Ratio.

In this section we would like to show that the following collineations we will introduce from [8] preserve cross-ratios. Now we start with giving the collineations, where $w,z,q,n \in \mathbf{A}$:

For any $u \notin \mathbf{I}$, the map L_u transforms points and lines as follows:

$$\begin{aligned}(x,y,1) &\to (ux,uyu,1)\\ (1,y,z\varepsilon) &\to \left(1,yu,(zu^{-1})\varepsilon\right)\\ (w\varepsilon,1,z\varepsilon) &\to \left((u^{-1}w)\varepsilon,1,(u^{-1}zu^{-1})\varepsilon\right)\end{aligned}$$

$$\begin{aligned}[m,1,k] &\to [mu,1,uku]\\ [1,n\varepsilon,p] &\to \left[1,(u^{-1}n)\varepsilon,up\right]\\ [q\varepsilon,n\varepsilon,1] &\to \left[(qu^{-1})\varepsilon,(u^{-1}nu^{-1})\varepsilon,1\right].\end{aligned}$$

For any $u \notin \mathbf{I}$, the map F_u transforms points and lines as follows:

$$\begin{aligned}(x,y,1) &\to (uxu,uy,1)\\ (1,y,z\varepsilon) &\to \left(1,u^{-1}y,(u^{-1}zu^{-1})\varepsilon\right)\\ (w\varepsilon,1,z\varepsilon) &\to \left((wu)\varepsilon,1,(zu^{-1})\varepsilon\right)\end{aligned}$$

$$\begin{aligned}[m,1,k] &\to \left[u^{-1}m,1,uk\right]\\ [1,n\varepsilon,p] &\to [1,(nu)\varepsilon,upu]\\ [q\varepsilon,n\varepsilon,1] &\to \left[(u^{-1}qu^{-1})\varepsilon,(nu^{-1})\varepsilon,1\right].\end{aligned}$$

For any $\alpha,\beta \in \mathbf{Z}(\mathcal{A})$, $\alpha,\beta \notin \mathbf{I}$, the map $S_{\alpha,\beta}$ transforms points and lines as follows:

$$\begin{aligned}(x,y,1) &\to (x\beta,y\alpha,1)\\ (1,y,z\varepsilon) &\to \left(1,\beta^{-1}y\alpha,(\beta^{-1}z)\varepsilon\right)\\ (w\varepsilon,1,z\varepsilon) &\to \left((\alpha^{-1}w\beta)\varepsilon,1,(\alpha^{-1}z)\varepsilon\right)\end{aligned}$$

$$\begin{aligned}[m,1,k] &\to \left[\beta^{-1}m\alpha,1,k\alpha\right]\\ [1,n\varepsilon,p] &\to \left[1,(\alpha^{-1}n\beta)\varepsilon,p\beta\right]\\ [q\varepsilon,n\varepsilon,1] &\to \left[(\beta^{-1}q)\varepsilon,(\alpha^{-1}n)\varepsilon,1\right].\end{aligned}$$

The map I_2 transforms points and lines as follows:

$$\begin{aligned}(x,y,1) &\to \left(y^{-1}x,y^{-1},1\right) \quad if \quad y\notin\mathbf{I}\\ (x,y,1) &\to \left(1,x^{-1},x^{-1}y\right) \quad if \quad y\in\mathbf{I}\ \wedge x\notin\mathbf{I}\\ (x,y,1) &\to (x,1,y) \quad if \quad y\in\mathbf{I}\ \wedge x\in\mathbf{I}\\ (1,y,z\varepsilon) &\to \left(y^{-1},\left(y^{-1}z\right)\varepsilon,1\right) \quad if \quad y\notin\mathbf{I}\\ (1,y,z\varepsilon) &\to (1,z\varepsilon,y) \quad if \quad y\in\mathbf{I}\\ (w\varepsilon,1,z\varepsilon) &\to (w\varepsilon,z\varepsilon,1)\end{aligned}$$

$$\begin{aligned}[m,1,k] &\to \left[-mk^{-1},1,k^{-1}\right] \quad if \quad k\notin\mathbf{I}\\ [m,1,k] &\to \left[1,-km^{-1},m^{-1}\right] \quad if \quad k\in\mathbf{I}\ \wedge m\notin\mathbf{I}\\ [m,1,k] &\to [m,k,1] \quad if \quad k\in\mathbf{I}\ \wedge m\in\mathbf{I}\\ [1,n\varepsilon,p] &\to \left[p^{-1},1,-\left(np^{-1}\right)\varepsilon\right] \quad if \quad p\notin\mathbf{I}\\ [1,n\varepsilon,p] &\to [1,p,n\varepsilon] \quad if \quad p\in\mathbf{I}\\ [q\varepsilon,n\varepsilon,1] &\to [q\varepsilon,1,n\varepsilon].\end{aligned}$$

Now we are ready to give the main result of the paper.

Theorem 3.1: The collineations L_u, F_u, $\mathrm{S}_{\alpha,\beta}$ and I_2 preserve cross-ratio.

Proof: Let A,B,C,D and Z be the points with the property given in the statement of Theorem 2.2. Then, it is obvious that

$$\begin{aligned}(A,B;C,D) &= (a,b;c,d) \qquad (1)\\ (Z,B;C,D) &= \left(z^{-1},b;c,d\right)\\ (A,Z;C,D) &= \left(a,z^{-1};c,d\right)\\ (A,B;Z,D) &= \left(a,b;z^{-1},d\right)\\ (A,B;C,Z) &= \left(a,b;c,z^{-1}\right),\end{aligned}$$

where $z \in \mathbf{I}$. In this case we must find the effect of φ to the points of any line where φ is any one of collineations L_u, F_u, $\mathrm{S}_{\alpha,\beta}$,and I_2.

i) Let $\varphi = \mathrm{L}_u$. If $l=[m,1,k]$, then

$$\begin{aligned}\varphi(X) &= \varphi(x,xm+k,1) = (ux,u(xm+k)u,1)\\ \varphi(Z) &= \varphi(1,m+zk,z) = \left(1,(m+zk)u,zu^{-1}\right)\end{aligned}$$

and $\varphi(l)=[mu,1,uku]$. From (a) of Theorem 2.2, we obtain

$$\begin{aligned}(\varphi(A),\varphi(B);\varphi(C),\varphi(D)) &= (ua,ub;uc,ud)\\ &=^{\sigma} (a,b;c,d)\\ (\varphi(Z),\varphi(B);\varphi(C),\varphi(D)) &= \left(uz^{-1},ub;uc,ud\right)\\ &=^{\sigma} \left(z^{-1},b;c,d\right),\end{aligned}$$

where $\sigma = l_{u^{-1}} \in \Lambda$.

If $l=[1,n,p]$, then

$$\begin{aligned}\varphi(X) &= \varphi(xn+p,x,1) = (u(xn+p),uxu,1)\\ \varphi(Z) &= \varphi(n+zp,1,z) = \left(u^{-1}(n+zp),1,u^{-1}zu^{-1}\right)\end{aligned}$$

and $\varphi(l)=\left[1,u^{-1}n,up\right]$. From (b) of Theorem 2.2, we have

$$\begin{aligned}(\varphi(A),\varphi(B);\varphi(C),\varphi(D)) &= (uau,ubu;ucu,udu)\\ &=^{\sigma} (a,b;c,d)\\ (\varphi(Z),\varphi(B);\varphi(C),\varphi(D)) &= \left(uz^{-1}u,ubu;ucu,udu\right)\\ &=^{\sigma} \left(z^{-1},b;c,d\right),\end{aligned}$$

where $\sigma = l_{u^{-1}} \circ r_{u^{-1}} \in \Lambda$.

If $l = [q, n, 1]$, then

$$\begin{aligned}\varphi(X) &= \varphi(1, x, q + xn) = \left(1, xu, (q + xn) u^{-1}\right) \\ \varphi(Z) &= \varphi(z, 1, zq + n) = \left(u^{-1}z, 1, u^{-1}(zq + n) u^{-1}\right)\end{aligned}$$

and $\varphi(l) = \left[qu^{-1}, u^{-1}nu^{-1}, 1\right]$. From (c) of Theorem 2.2, we obtain

$$\begin{aligned}(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (au, bu; cu, du) \\ &=^{\sigma} (a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(z^{-1}u, bu; cu, du\right) \\ &=^{\sigma} \left(z^{-1}, b; c, d\right),\end{aligned}$$

where $\sigma = r_{u^{-1}} \in \Lambda$.

ii) Let $\varphi = \mathrm{F}_u$. If $l = [m, 1, k]$, then

$$\begin{aligned}\varphi(X) &= \varphi(x, xm + k, 1) = (uxu, u(xm + k), 1) \\ \varphi(Z) &= \varphi(1, m + zk, z) = \left(1, u^{-1}(m + zk), u^{-1}zu^{-1}\right)\end{aligned}$$

and $\varphi(l) = \left[u^{-1}m, 1, uk\right]$. From (a) of Theorem 2.2, we have

$$\begin{aligned}(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (uau, ubu; ucu, udu) \\ &=^{\sigma} (a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(uz^{-1}u, ubu; ucu, udu\right) \\ &=^{\sigma} \left(z^{-1}, b; c, d\right),\end{aligned}$$

where $\sigma = l_{u^{-1}} \circ r_{u^{-1}} \in \Lambda$.

If $l = [1, n, p]$, then

$$\begin{aligned}\varphi(X) &= \varphi(xn + p, x, 1) = (u(xn + p) u, ux, 1) \\ \varphi(Z) &= \varphi(n + zp, 1, z) = \left((n + zp) u, 1, zu^{-1}\right)\end{aligned}$$

and $\varphi(l) = [1, nu, upu]$. From (b) of Theorem 2.2, we obtain

$$\begin{aligned}(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (ua, ub; uc, ud) \\ &=^{\sigma} (a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(uz^{-1}, ub; uc, ud\right) \\ &=^{\sigma} \left(z^{-1}, b; c, d\right),\end{aligned}$$

where $\sigma = l_{u^{-1}} \in \Lambda$.

If $l = [q, n, 1]$, then

$$\begin{aligned}\varphi(X) &= \varphi(1, x, q + xn) = \left(1, u^{-1}x, u^{-1}(q + xn) u^{-1}\right) \\ \varphi(Z) &= \varphi(z, 1, zq + n) = \left(zu, 1, (zq + n) u^{-1}\right)\end{aligned}$$

and $\varphi(l) = \left[u^{-1}qu^{-1}, nu^{-1}, 1\right]$. From (c) of Theorem 2.2, we have

$$\begin{aligned}&(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) \\ &= \left(u^{-1}a, u^{-1}b; u^{-1}c, u^{-1}d\right) =^{\sigma} (a, b; c, d) \\ &(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) \\ &= \left(u^{-1}z^{-1}, u^{-1}b; u^{-1}c, u^{-1}d\right) =^{\sigma} \left(z^{-1}, b; c, d\right),\end{aligned}$$

where $\sigma = l_u \in \Lambda$.

iii) Let $\varphi = \mathrm{S}_{\alpha,\beta}$. If $l = [m, 1, k]$, , then

$$\begin{aligned}\varphi(X) &= \varphi(x, xm + k, 1) = (x\beta, (xm + k) \alpha, 1) \\ \varphi(Z) &= \varphi(1, m + zk, z) = \left(1, \beta^{-1}(m + zk) \alpha, \beta^{-1}z\right)\end{aligned}$$

and $\varphi(l) = \left[\beta^{-1}m\alpha, 1, k\alpha\right]$. From (a) of Theorem 2.2, we obtain

$$\begin{aligned}(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a\beta, b\beta; c\beta, d\beta) \\ &=^{\sigma} (a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(z^{-1}\beta, b\beta; c\beta, d\beta\right) \\ &=^{\sigma} \left(z^{-1}, b; c, d\right),\end{aligned}$$

where $\sigma = r_{\beta^{-1}} \in \Lambda$.

If $l = [1, n, p]$, then

$$\begin{aligned}\varphi(X) &= \varphi(xn + p, x, 1) = ((xn + p) \beta, x\alpha, 1) \\ \varphi(Z) &= \varphi(n + zp, 1, z) = \left(\alpha^{-1}(n + zp) \beta, 1, \alpha^{-1}z\right)\end{aligned}$$

and $\varphi(l) = \left[1, \alpha^{-1}n\beta, p\beta\right]$. From (b) of Theorem 2.2, we have

$$\begin{aligned}(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a\alpha, b\alpha; c\alpha, d\alpha) \\ &=^{\sigma} (a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(z^{-1}\alpha, b\alpha; c\alpha, d\alpha\right) \\ &=^{\sigma} \left(z^{-1}, b; c, d\right),\end{aligned}$$

where $\sigma = r_{\alpha^{-1}} \in \Lambda$.

If $l = [q, n, 1]$, then

$$\begin{aligned}\varphi(X) &= \varphi(1, x, q + xn) = \left(1, \beta^{-1}x\alpha, \beta^{-1}(q + xn)\right) \\ \varphi(Z) &= \varphi(z, 1, zq + n) = \left(\alpha^{-1}z\beta, 1, \alpha^{-1}(zq + n)\right)\end{aligned}$$

and $\varphi(l) = \left[\beta^{-1}q, \alpha^{-1}n, 1\right]$. From (c) of Theorem 2.2, we obtain

$$\begin{aligned}&(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) \\ &= \left(\beta^{-1}a\alpha, \beta^{-1}b\alpha; \beta^{-1}c\alpha, \beta^{-1}d\alpha\right) =^{\sigma} (a, b; c, d) \\ &(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) \\ &= \left(\beta^{-1}z^{-1}\alpha, \beta^{-1}b\alpha; \beta^{-1}c\alpha, \beta^{-1}d\alpha\right) =^{\sigma} \left(z^{-1}, b; c, d\right),\end{aligned}$$

where $\sigma = l_{\beta} \circ r_{\alpha^{-1}} \in \Lambda$.

iv) Let $\varphi = \mathrm{I}_2$. If $l = [m, 1, k]$, then

$$\begin{aligned}\varphi(X) &= \varphi(x, xm + k, 1) \\ &= \left((xm + k)^{-1} x, (xm + k)^{-1}, 1\right), \\ &\quad where\ xm + k \notin \mathbf{I} \\ \varphi(X) &= \varphi(x, xm + k, 1) \\ &= \left(1, x^{-1}, x^{-1}(xm + k)\right), \\ &\quad where\ xm + k \in \mathbf{I}\ and\ x \notin \mathbf{I} \\ \varphi(X) &= \varphi(x, xm + k, 1) \\ &= (x, 1, xm + k),\ where\ xm + k \in \mathbf{I}\ and\ x \in \mathbf{I} \\ \varphi(Z) &= \varphi(1, m + zk, z) \\ &= \left((m + zk)^{-1}, (m + zk)^{-1} z, 1\right), \\ &\quad where\ m + zk \notin \mathbf{I} \\ \varphi(Z) &= \varphi(1, m + zk, z) \\ &= (1, z, m + zk),\ where\ m + zk \in \mathbf{I}\end{aligned}$$

and

$$\begin{aligned}\varphi(l) &= \left[-mk^{-1},1,k^{-1}\right],\ where\ k\notin \mathbf{I}\\ \varphi(l) &= \left[1,-km^{-1},m^{-1}\right],\ where\ k\in \mathbf{I}\ and\ m\notin \mathbf{I}\\ \varphi(l) &= [m,k,1],\ where\ k\in \mathbf{I}\ and\ m\in \mathbf{I}.\end{aligned}$$

In this case, from (a) of Theorem 2.2, the cross-ratio of the points of $\left[-mk^{-1},1,k^{-1}\right]$ is as follows:

$$\begin{aligned}&(\varphi(A),\varphi(B);\varphi(C),\varphi(D))\\ &=\left((am+k)^{-1}a,(bm+k)^{-1}b;\right.\\ &\left.(cm+k)^{-1}c,(dm+k)^{-1}d\right)\\ &=^{\sigma}(a,b;c,d)\end{aligned}$$

$$\begin{aligned}&(\varphi(Z),\varphi(B);\varphi(C),\varphi(D))\\ &=\left((m+zk)^{-1},(bm+k)^{-1}b;\right.\\ &\left.(cm+k)^{-1}c,(dm+k)^{-1}d\right)\\ &=^{\sigma}\left(z^{-1},b;c,d\right),\end{aligned}$$

where $\sigma=i\circ r_{k^{-1}}\circ t_{-m}\circ i\in\Lambda$. From (b) of Theorem 2.2, the cross-ratio of the points of $\left[1,-km^{-1},m^{-1}\right]$ is as follows:

$$\begin{aligned}&(\varphi(A),\varphi(B);\varphi(C),\varphi(D))\\ &=\left((am+k)^{-1},(bm+k)^{-1};\right.\\ &\left.(cm+k)^{-1},(dm+k)^{-1}\right)\\ &=^{\sigma}(a,b;c,d)\end{aligned}$$

$$\begin{aligned}&(\varphi(Z),\varphi(B);\varphi(C),\varphi(D))\\ &=\left((m+zk)^{-1}z,(bm+k)^{-1};\right.\\ &\left.(cm+k)^{-1},(dm+k)^{-1}\right)\\ &=^{\sigma}\left(z^{-1},b;c,d\right),\end{aligned}$$

where $\sigma=r_{m^{-1}}\circ t_{-k}\circ i\in\Lambda$. From (c) of Theorem 2.2, the cross-ratio of the points of $[m,k,1]$ is as follows:

$$\begin{aligned}(\varphi(A),\varphi(B);\varphi(C),\varphi(D)) &= \left(a^{-1},b^{-1};c^{-1},d^{-1}\right)\\ &= {}^{\sigma}(a,b;c,d)\\ (\varphi(Z),\varphi(B);\varphi(C),\varphi(D)) &= \left(z,b^{-1};c^{-1},d^{-1}\right)\\ &= {}^{\sigma}\left(z^{-1},b;c,d\right),\end{aligned}$$

where $\sigma=i\in\Lambda$.

If $l=[1,n,p]$, then

$$\begin{aligned}\varphi(X) &= \varphi(xn+p,x,1)\\ &= \left(x^{-1}(xn+p),x^{-1},1\right),\ where\ x\notin \mathbf{I}\\ \varphi(X) &= \varphi(xn+p,x,1)\\ &= \left(1,(xn+p)^{-1},(xn+p)^{-1}x\right),\\ &\quad where\ x\in \mathbf{I}\ and\ xn+p\notin \mathbf{I}\\ \varphi(X) &= \varphi(xn+p,x,1)\\ &= (xn+p,1,x),\ where\ x\in \mathbf{I}\ and\ xn+p\in \mathbf{I}\\ \varphi(Z) &= \varphi(n+zp,1,z)=(n+zp,z,1)\end{aligned}$$

and

$$\begin{aligned}\varphi(l) &= \left[p^{-1},1,-np^{-1}\right],\ where\ p\notin \mathbf{I}\\ \varphi(l) &= [1,p,n],\ where\ p\in \mathbf{I}.\end{aligned}$$

In this case, from (a) of Theorem 2.2, the cross-ratio of the points of $\left[p^{-1},1,-np^{-1}\right]$ is as follows:

$$\begin{aligned}(\varphi(A),\varphi(B);\varphi(C),\varphi(D)) &= \left(a^{-1}(an+p),b^{-1}(bn+p);\right.\\ &\quad\left.c^{-1}(cn+p),d^{-1}(dn+p)\right)\\ &=^{\sigma}(a,b;c,d)\\ (\varphi(Z),\varphi(B);\varphi(C),\varphi(D)) &= \left(n+zp,b^{-1}(bn+p);\right.\\ &\quad\left.c^{-1}(cn+p),d^{-1}(dn+p)\right)\\ &=^{\sigma}\left(z^{-1},b;c,d\right),\end{aligned}$$

where $\sigma=i\circ r_{p^{-1}}\circ t_{-n}\in\Lambda$. From (b) of Theorem 2.2, the cross-ratio of the points of $[1,p,n]$ is as follows:

$$\begin{aligned}(\varphi(A),\varphi(B);\varphi(C),\varphi(D)) &= \left(a^{-1},b^{-1};c^{-1},d^{-1}\right)\\ &=^{\sigma}(a,b;c,d)\\ (\varphi(Z),\varphi(B);\varphi(C),\varphi(D)) &= \left(z,b^{-1};c^{-1},d^{-1}\right)\\ &=^{\sigma}\left(z^{-1},b;c,d\right),\end{aligned}$$

where $\sigma=i\in\Lambda$.

If $l=[q,n,1]$, then

$$\begin{aligned}\varphi(X) &= \varphi(1,x,q+xn)\\ &= \left(x^{-1},x^{-1}(q+xn),1\right),\ where\ x\notin \mathbf{I}\\ \varphi(X) &= \varphi(1,x,q+xn)\\ &= (1,q+xn,x),\ where\ x\in \mathbf{I}\\ \varphi(Z) &= \varphi(z,1,zq+n)=(z,zq+n,1)\end{aligned}$$

and $\varphi(l)=[q,1,n]$. In this case, from (a) of Theorem 2.2, the cross-ratio of the points of $[q,1,n]$ is as follows:

$$\begin{aligned}(\varphi(A),\varphi(B);\varphi(C),\varphi(D)) &= \left(a^{-1},b^{-1};c^{-1},d^{-1}\right)\\ &=^{\sigma}(a,b;c,d)\\ (\varphi(Z),\varphi(B);\varphi(C),\varphi(D)) &= \left(z,b^{-1};c^{-1},d^{-1}\right)\\ &=^{\sigma}\left(z^{-1},b;c,d\right),\end{aligned}$$

where $\sigma=i\in\Lambda$.

Consequently, by considering other all cases we get

$$\begin{aligned}(\varphi(A),\varphi(B);\varphi(C),\varphi(D)) &= (a,b;c,d)\\ (\varphi(Z),\varphi(B);\varphi(C),\varphi(D)) &= \left(z^{-1},b;c,d\right)\\ (\varphi(A),\varphi(Z);\varphi(C),\varphi(D)) &= \left(a,z^{-1};c,d\right)\\ (\varphi(A),\varphi(B);\varphi(Z),\varphi(D)) &= \left(a,b;z^{-1},d\right)\\ (\varphi(A),\varphi(B);\varphi(C),\varphi(Z)) &= \left(a,b;c,z^{-1}\right)\end{aligned}$$

for every collineation φ. Combining the last result and the result of (1), the proof is completed. ■

Remark 3.2: In the present paper we show that the collineations L_u, F_u, $S_{\alpha,\beta}$,and I_2, given in [8], preserve cross-ratio. *A paper related to* the result that the other collineations of [8] ($T_{u,v}$, I_1, F and G_u) preserve cross-ratio, is under review.

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