

A Method to Calculate Frenet Apparatus of W-Curves in the Euclidean 6-Space

Süha Yılmaz and Melih Turgut

Abstract—These In this work, a regular unit speed curve in six dimensional Euclidean space, whose Frenet curvatures are constant, is considered. Thereafter, a method to calculate Frenet apparatus of this curve is presented.

Keywords—Classical Differential Geometry, Euclidean 6-space, Frenet Apparatus of the curves.

I. INTRODUCTION

IT is safe to report that the many important results in the theory of the curves in E^3 were initiated by G. Monge; and G. Darboux pioneered the moving frame idea. Thereafter, F. Frenet defined his moving frame and his special equations which play important role in mechanics and kinematics as well as in differential geometry (for more details see [2]). At the beginning of the twentieth century, A. Einstein's theory opened a door of use of new geometries with respect to physical meanings. These geometries mostly have higher dimensions. In higher dimensional Euclidean space, researchers treated some of classical differential geometry topics [3], [4] and [7].

It is well-known that, if a curve differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve, is called *Frenet apparatus* of the curves. In [6] and [8], authors presented a method to determine Frenet apparatus of the curves in E^4 and E^5 in analogy with the method in E^3 .

In this work, using vector product defined in [1], the method in E^6 is expressed. Thus, this classical topic is extended to the space E^6 . This is one of the first study in the Euclidean 6-space in classical manner.

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II. PRELIMINARIES

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E^6 are briefly presented. (A more complete elementary treatment can be found in [4].)

Let $\vec{\alpha} : I \subset R \rightarrow E^6$ be an arbitrary curve in the Euclidean space E^6 . Recall that the curve $\vec{\alpha}$ is said to be of unit speed (or parameterized by arclength function s) if $\langle \vec{\alpha}', \vec{\alpha}' \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the standard scalar (inner) product of E^6 given by

$$\langle \vec{a}, \vec{b} \rangle = \sum_{i=1}^6 a_i b_i \tag{1}$$

for each $\vec{a} = (a_1, a_2, a_3, a_4, a_5, a_6)$, $\vec{b} = (b_1, b_2, b_3, b_4, b_5, b_6) \in E^6$. In particular, the norm of a vector $\vec{a} \in E^6$ is given by $\|\vec{a}\| = \sqrt{\langle \vec{a}, \vec{a} \rangle}$.

Denote by $\{\vec{V}_1(s), \vec{V}_2(s), \vec{V}_3(s), \vec{V}_4(s), \vec{V}_5(s), \vec{V}_6(s)\}$ the moving Frenet frame along the unit speed curve $\vec{\alpha}$. Then the Frenet formulas are given by (see [4])

$$\begin{bmatrix} \vec{V}'_1 \\ \vec{V}'_2 \\ \vec{V}'_3 \\ \vec{V}'_4 \\ \vec{V}'_5 \\ \vec{V}'_6 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & k_3 & 0 & 0 \\ 0 & 0 & -k_3 & 0 & k_4 & 0 \\ 0 & 0 & 0 & -k_4 & 0 & k_5 \\ 0 & 0 & 0 & 0 & -k_5 & 0 \end{bmatrix} \begin{bmatrix} \vec{V}_1 \\ \vec{V}_2 \\ \vec{V}_3 \\ \vec{V}_4 \\ \vec{V}_5 \\ \vec{V}_6 \end{bmatrix} \tag{2}$$

The functions $k_1(s), k_2(s), k_3(s), k_4(s)$ and $k_5(s)$ are called, respectively, the first, the second, the third, the fourth and the fifth curvature of the curve $\vec{\alpha}$. If $k_5(s) \neq 0$ for each $s \in I \subset R$, the curve $\vec{\alpha}$ lies fully in E^6 . Recall that the unit sphere S^5 in E^6 , centered at the origin, is the hypersurface defined by

$$S^5 = \{ \vec{x} \in E^6 : \langle \vec{x}, \vec{x} \rangle = 1 \}. \tag{3}$$

In [1], with an analogous way in Euclidean 3-space, the author defines a vector product in E^6 with the following definition.

Definition 1 Let $\vec{a} = (a_1, a_2, \dots, a_6)$, $\vec{b} = (b_1, b_2, \dots, b_6)$, $\vec{c} = (c_1, c_2, \dots, c_6)$, $\vec{d} = (d_1, d_2, \dots, d_6)$ and $\vec{f} = (f_1, f_2, \dots, f_6)$ be vectors in E^6 . The vector product of E^6 is defined with the determinant

$$\vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} \wedge \vec{f} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 & \vec{e}_5 & \vec{e}_6 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \end{vmatrix},$$

where \vec{e}_i for $1 \leq i \leq 6$ are coordinate direction (basis) vectors of E^6 which satisfy

$$\begin{aligned} \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 \wedge \vec{e}_4 \wedge \vec{e}_5 &= \vec{e}_6, \quad \vec{e}_2 \wedge \vec{e}_3 \wedge \vec{e}_4 \wedge \vec{e}_5 \wedge \vec{e}_6 = \vec{e}_1, \\ \vec{e}_3 \wedge \vec{e}_4 \wedge \vec{e}_5 \wedge \vec{e}_6 \wedge \vec{e}_1 &= \vec{e}_2, \quad \vec{e}_4 \wedge \vec{e}_5 \wedge \vec{e}_6 \wedge \vec{e}_1 \wedge \vec{e}_2 = \vec{e}_3, \\ \vec{e}_5 \wedge \vec{e}_6 \wedge \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 &= \vec{e}_4. \end{aligned}$$

Via this product it is safe to report that

$$\begin{aligned} \langle \vec{a}, \vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} \wedge \vec{f} \rangle &= \langle \vec{b}, \vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} \wedge \vec{f} \rangle \\ &= \langle \vec{c}, \vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} \wedge \vec{f} \rangle \\ &= \langle \vec{d}, \vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} \wedge \vec{f} \rangle \\ &= \langle \vec{f}, \vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} \wedge \vec{f} \rangle = 0. \end{aligned}$$

And here, recall that a regular curve $\vec{\varphi}$ is called a W -curve, if it has constant Frenet curvatures [5].

III. THE METHOD

Let $\vec{X} = \vec{X}(s)$ be an unit speed W -curve in E^6 . Our aim is to determine formulas of the Frenet apparatus. First we write the following derivatives according to s .

$$\vec{X}' = \vec{V}_1. \tag{4}$$

$$\vec{X}'' = k_1 \vec{V}_2. \tag{5}$$

$$\vec{X}''' = -k_1^2 \vec{V}_1 + k_1 k_2 \vec{V}_3. \tag{6}$$

$$\vec{X}^{(IV)} = -(k_1^3 + k_1 k_2^2) \vec{V}_2 + k_1 k_2 k_3 \vec{V}_4. \tag{7}$$

$$\vec{X}^{(V)} = k_1^2 (k_1^2 + k_2^2) \vec{V}_1 - k_1 k_2 (k_1^2 + k_2^2) \vec{V}_3 + k_1 k_2 k_3 k_4 \vec{V}_5. \tag{8}$$

$$\begin{aligned} \vec{X}^{(VI)} &= (\dots) \vec{V}_1 + (\dots) \vec{V}_2 + (\dots) \vec{V}_3 \\ &+ (\dots) \vec{V}_4 + (\dots) \vec{V}_5 + k_1 k_2 k_3 k_4 k_5 \vec{V}_6. \end{aligned} \tag{9}$$

Taking the norm of both sides of (5), we have the first curvature as

$$\|\vec{X}''\| = k_1(s). \tag{10}$$

And therefore, we obtain \vec{V}_2

$$\vec{V}_2 = \frac{\vec{X}''}{k_1}. \tag{11}$$

Then, taking the norm of both sides of (6), we write the second curvature as

$$k_2 = \frac{\sqrt{\|\vec{X}'''\|^2 - \|\vec{X}''\|^4}}{\|\vec{X}''\|}. \tag{12}$$

Using (12) and considering (6), we have the third vector field

$$\vec{V}_3 = \frac{\vec{X}''' + \|\vec{X}''\|^2 \cdot \vec{X}'}{\sqrt{\|\vec{X}'''\|^2 - \|\vec{X}''\|^4}}. \tag{13}$$

To calculate the third curvature and the fourth vector field, let us form

$$\|\vec{X}'''\|^2 \cdot \vec{X}^{(IV)} + \|\vec{X}'''\|^2 \cdot \vec{X}'' = k_1^3 k_2 k_3 \vec{V}_4. \tag{14}$$

Equation (14) yields, respectively,

$$\vec{V}_4 = \frac{\|\vec{X}'''\|^2 \cdot \vec{X}^{(IV)} + \|\vec{X}'''\|^2 \cdot \vec{X}''}{\|\|\vec{X}'''\|^2 \cdot \vec{X}^{(IV)} + \|\vec{X}'''\|^2 \cdot \vec{X}''\|}, \tag{15}$$

$$k_3 = \frac{\|\|\vec{X}'''\|^2 \cdot \vec{X}^{(IV)} + \|\vec{X}'''\|^2 \cdot \vec{X}''\|}{\|\vec{X}'''\|^2 \cdot \sqrt{\|\vec{X}'''\|^2 - \|\vec{X}''\|^4}}. \tag{16}$$

Now, let us calculate vector product of $\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{X}''' \wedge \vec{X}^{(IV)} \wedge \vec{X}^{(V)}$ according to frame $\{\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4, \vec{V}_5, \vec{V}_6\}$. This expression follows that

$$\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{X}''' \wedge \vec{X}^{(IV)} \wedge \vec{X}^{(V)} = -k_1^3 k_2^2 k_3^2 k_4 \vec{V}_5. \tag{17}$$

Using (17), we easily have \vec{V}_6 and the fourth curvature of the curve $\vec{X} = \vec{X}(s)$, respectively,

$$\vec{V}_6 = \eta \frac{\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{X}''' \wedge \vec{X}^{(IV)} \wedge \vec{X}^{(V)}}{\|\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{X}''' \wedge \vec{X}^{(IV)} \wedge \vec{X}^{(V)}\|}, \quad (18)$$

$$k_4 = \frac{\|\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{X}''' \wedge \vec{X}^{(IV)} \wedge \vec{X}^{(V)}\| \|\vec{X}''\|^3}{\|\|\vec{X}''\|^2 \cdot \vec{X}^{(IV)} + \|\vec{X}'''\|^2 \cdot \vec{X}''\|^2}. \quad (19)$$

η in the expression (18) is taken ± 1 to make +1 determinant of $[\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4, \vec{V}_5, \vec{V}_6]$ matrix. By this way, Frenet frame is positively oriented. Using the inner product of (9) and (18), we obtain $\langle \vec{X}^{(VI)}, \vec{V}_6 \rangle = k_1 k_2 k_3 k_4 k_5$. Since the fifth curvature of $\vec{X}(s)$ as

$$k_5 = \frac{\langle \vec{X}^{(VI)}, \vec{V}_6 \rangle \|\|\vec{X}''\|^2 \cdot \vec{X}^{(IV)} + \|\vec{X}'''\|^2 \cdot \vec{X}''\|}{\|\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{X}''' \wedge \vec{X}^{(IV)} \wedge \vec{X}^{(V)}\| \|\vec{X}''\|}. \quad (20)$$

And, finally, the vector product

$$\vec{V}_5 = \eta \cdot \vec{V}_3 \wedge \vec{V}_2 \wedge \vec{V}_1 \wedge \vec{V}_5 \wedge \vec{V}_6 \quad (21)$$

gives us fifth vector field of the Frenet frame. Thus, we have calculated Frenet apparatus of the curve $\vec{X} = \vec{X}(s)$. Moreover, suffice it to say that $\{\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4, \vec{V}_5, \vec{V}_6\}$ is an orthonormal frame of E^6 .

IV. CONCLUSION AND FURTHER REMARKS

Throughout the presented paper, one of classical topic in the theory of the curves in E^6 was treated. In the recent papers, although Frenet frame vectors and curvatures are defined, there was not a method to calculate all Frenet apparatus of an unit speed W -curve which lies fully in E^6 . Here, using vector product, we give formulas of frame vectors (and therefore curvatures).

In this paper, we cannot generalize and extend the method to the space E^n . Because, it has been observed that consecutive derivatives of the position vector of a regular curve cannot be formed by a summation formula. It goes on unsystematic. Moreover, it has been seen that, in the Euclidean space E^n the first and the last curvature of the curve can be easily calculated. But, determination of i^{th} curvatures and frame vectors for $1 < i < n$ is not clear. Since, we deal with the curves in E^6 .

Via this method, some of classical differential geometry topics can be treated. Relations among spherical indicators, Bertrand curves and Involute-evolute curve couple may be easily calculated. And it is well-known that ID -Module, which is used for kinematical applications and robotics, is congruent to space E^6 . Also, the presented method may be

used such modelling processes.

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