

Quadratic Irrationals, Quadratic Ideals and Indefinite Quadratic Forms II

Ahmet Tekcan, Arzu Özkoç

Abstract—Let $D \neq 1$ be a positive non-square integer and let $\delta = \sqrt{D}$ or $\frac{1+\sqrt{D}}{2}$ be a real quadratic irrational with trace $t = \delta + \bar{\delta}$ and norm $n = \delta\bar{\delta}$. Let $\gamma = \frac{P+\delta}{Q}$ be a quadratic irrational for positive integers P and Q . Given a quadratic irrational γ , there exist a quadratic ideal $I_\gamma = [Q, \delta + P]$ and an indefinite quadratic form $F_\gamma(x, y) = Q(x - \gamma y)(x - \bar{\gamma}y)$ of discriminant $\Delta = t^2 - 4n$. In the first section, we give some preliminaries form binary quadratic forms, quadratic irrationals and quadratic ideals. In the second section, we obtain some results on γ , I_γ and F_γ for some specific values of Q and P .

Keywords—Quadratic irrationals, quadratic ideals, indefinite quadratic forms, extended modular group.

I. PRELIMINARIES.

A real quadratic form (or just a form) F is a polynomial in two variables x, y of the type

$$F = F(x, y) = ax^2 + bxy + cy^2 \quad (1)$$

with real coefficients a, b, c . We denote F briefly by

$$F = (a, b, c).$$

The discriminant of F is defined by the formula $b^2 - 4ac$ and is denoted by Δ . Moreover F is an integral form if and only if $a, b, c \in \mathbf{Z}$ and F is indefinite if and only if $\Delta > 0$.

Let Γ be the modular group $\text{PSL}(2, \mathbf{Z})$, i.e. the set of the transformations

$$z \mapsto \frac{rz + s}{tz + u}, \quad r, s, t, u \in \mathbf{Z}, \quad ru - st = 1.$$

Then Γ is generated by the transformations $T(z) = \frac{-1}{z}$ and $V(z) = z + 1$. Let $U = T.V$. Then $U(z) = \frac{-1}{z+1}$. Then Γ has a representation $\Gamma = \langle T, U : T^2 = U^3 = I \rangle$. So

$$\Gamma = \left\{ g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} : r, s, t, u \in \mathbf{Z}, ru - st = 1 \right\}. \quad (2)$$

We denote the symmetry with respect to the imaginary axis with R , that is $R(z) = -\bar{z}$. Then the group $\bar{\Gamma} = \Gamma \cup R\Gamma$ is generated by the transformations R, T, U and has a representation $\bar{\Gamma} = \langle R, T, U : R^2 = T^2 = U^3 = I \rangle$, and is called the extended modular group. So

$$\bar{\Gamma} = \left\{ g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} : r, s, t, u \in \mathbf{Z}, ru - st = \pm 1 \right\}. \quad (3)$$

There is a strong connection between the extended modular group and binary quadratic forms (see [5]). Most properties of

Ahmet Tekcan and Arzu Özkoç are with the Uludag University, Department of Mathematics, Faculty of Science, Bursa-TURKIYE. emails: tekcan@uludag.edu.tr, aozkoc@uludag.edu.tr http://matematik.uludag.edu.tr/AhmetTekcan.htm.

binary quadratic forms can be given by the aid of the extended modular group. Gauss (1777-1855) defined the group action of $\bar{\Gamma}$ on the set of forms as follows: Let $F = (a, b, c)$ be a quadratic form and let $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \bar{\Gamma}$. Then the form gF is defined by

$$gF(x, y) = (ar^2 + brs + cs^2)x^2 + (2art + brs + bts + 2csu)xy + (at^2 + btu + cu^2)y^2, \quad (4)$$

that is, gF is gotten from F by making the substitution

$$x \rightarrow rx + tu, \quad y \rightarrow sx + uy.$$

Moreover, $\Delta(F) = \Delta(gF)$ for all $g \in \bar{\Gamma}$, that is, the action of $\bar{\Gamma}$ on forms leaves the discriminant invariant. If F is indefinite or integral, then so is gF for all $g \in \bar{\Gamma}$.

Let F and G be two forms. If there exists a $g \in \bar{\Gamma}$ such that $gF = G$, then F and G are called equivalent. If $\det g = 1$, then F and G are called properly equivalent and if $\det g = -1$, then F and G are called improperly equivalent. A quadratic form F is said to be ambiguous if it is improperly equivalent to itself. An indefinite quadratic form F of discriminant Δ is said to be reduced if

$$|\sqrt{\Delta} - 2|a|| < b < \sqrt{\Delta}. \quad (5)$$

Mollin (see [1]) considered the arithmetic of ideals in his book. Let $D \neq 1$ be a square free integer and let $\Delta = \frac{4D}{r^2}$, where

$$r = \begin{cases} 2 & D \equiv 1 \pmod{4} \\ 1 & \text{otherwise.} \end{cases} \quad (6)$$

If we set $\mathbf{K} = \mathbf{Q}(\sqrt{D})$, then \mathbf{K} is called a quadratic number field of discriminant $\Delta = \frac{4D}{r^2}$. A complex number is an algebraic integer if it is the root of a monic polynomial with coefficients in \mathbf{Z} . The set of all algebraic integers in the complex field \mathbf{C} is a ring which we denote by A . Therefore $A \cap \mathbf{K} = O_\Delta$ is the ring of integers of the quadratic field \mathbf{K} of discriminant Δ . Set

$$w_\Delta = \frac{r-1+\sqrt{D}}{r}$$

for r defined in (6). Then w_Δ is called principal surd. We restate the ring of integers of \mathbf{K} as

$$O_\Delta = [1, w_\Delta] = \mathbf{Z}[w_\Delta].$$

In this case $\{1, w_\Delta\}$ is called an integral basis for \mathbf{K} . Let $I = [\alpha, \beta]$ denote the \mathbf{Z} -module $\alpha\mathbf{Z} \oplus \beta\mathbf{Z}$, i.e., the additive

abelian group, with basis elements α and β consisting of

$$\{\alpha x + \beta y : x, y \in \mathbf{Z}\}.$$

Note that $O_\Delta = \left[1, \frac{1+\sqrt{\Delta}}{r}\right]$. In this case $w_\Delta = \frac{r-1+\sqrt{\Delta}}{r}$ is called the principal surd. Every principal surd $w_\Delta \in O_\Delta$ can be uniquely expressed as

$$w_\Delta = x\alpha + y\beta,$$

where $x, y \in \mathbf{Z}$ and $\alpha, \beta \in O_\Delta$. We call α, β an integral basis for \mathbf{K} , and denote it by $[\alpha, \beta]$. If $\frac{\alpha\bar{\beta}-\beta\bar{\alpha}}{\sqrt{\Delta}} > 0$, then α and β are called ordered basis elements. Recall that two basis of an ideal are ordered if and only if they are equivalent under an element of $\bar{\Gamma}$. If I has ordered basis elements, then we say that I is simply ordered. If I is ordered, then

$$F(x, y) = \frac{N(\alpha x + \beta y)}{N(I)}$$

is a quadratic form of discriminant Δ (Here $N(x)$, denote the norm of x). In this case we say that F belongs to I and write $I \rightarrow F$.

Conversely let us assume that

$$G(x, y) = Ax^2 + Bxy + Cy^2 = d(ax^2 + bxy + cy^2)$$

be a quadratic form, where $d = \pm gcd(A, B, C)$ and $b^2 - 4ac = \Delta$. If $B^2 - 4AC > 0$, then we get $d > 0$ and if $B^2 - 4AC < 0$, then we choose d such that $a > 0$. Set

$$I = [\alpha, \beta] = \left[a, \frac{b - \sqrt{\Delta}}{2} \right]$$

for $a > 0$ or

$$I = [\alpha, \beta] = \left[a, \frac{b - \sqrt{\Delta}}{2} \right] \sqrt{\Delta}$$

for $a < 0$ and $\Delta > 0$. Then I is an ordered O_Δ -ideal. Thus to every form G , there corresponds an ideal I to which G belongs and we write $G \rightarrow I$. Hence we have a correspondence between ideals and quadratic forms (for further details see [2], [3], [4], [7]).

Theorem 1.1: If $I = [a, b + cw_\Delta]$, then I is a non-zero ideal of O_Δ if and only if $c|b, c|a$ and $ac|N(b + cw_\Delta)$ [1].

Let δ denote a real quadratic irrational integer with trace $t = \delta + \bar{\delta}$ and norm $n = \delta\bar{\delta}$. Given a real quadratic irrational $\gamma \in \mathbf{Q}(\delta)$, there are rational integers P and Q such that $\gamma = \frac{P+\delta}{Q}$ with $Q|(\delta + P)(\bar{\delta} + P)$. Hence for each

$$\gamma = \frac{P + \delta}{Q}, \tag{7}$$

there is a corresponding \mathbf{Z} -module

$$I_\gamma = [Q, P + \delta] \tag{8}$$

in fact, this module is an ideal by Theorem 1.1. The conjugate of I_γ is defined as

$$\bar{I}_\gamma = [Q, P + \bar{\delta}].$$

If $I_\gamma = \bar{I}_\gamma$, then I_γ is called ambiguous. The ideal I_γ in (8) is said to be reduced if and only if

$$P + \delta > Q \text{ and } -Q < P + \bar{\delta} < 0. \tag{9}$$

So I_γ is ambiguous if and only if it contains both $\frac{P+\delta}{Q}$ and $\frac{P+\bar{\delta}}{Q}$, so if and only if

$$\frac{2P}{Q} \in \mathbf{Z}.$$

For the quadratic irrational γ , there exists an indefinite quadratic form

$$F_\gamma(x, y) = Q(x - \gamma y)(x - \bar{\gamma} y). \tag{10}$$

Applying (10), we obtain

$$\begin{aligned} F_\gamma(x, y) &= Q(x - \gamma y)(x - \bar{\gamma} y) \\ &= Q [x^2 - xy(\gamma + \bar{\gamma}) + y^2(\gamma\bar{\gamma})] \\ &= Q \left[x^2 - xy \left(\frac{P+\delta}{Q} + \frac{P+\bar{\delta}}{Q} \right) + y^2 \left(\frac{P+\delta}{Q} \cdot \frac{P+\bar{\delta}}{Q} \right) \right] \\ &= Q \left[x^2 - xy \left(\frac{t+2P}{Q} \right) + y^2 \left(\frac{P^2 + P(\delta+\bar{\delta}) + \delta\bar{\delta}}{Q} \right) \right] \\ &= Q \left[x^2 - xy \left(\frac{t+2P}{Q} \right) + y^2 \left(\frac{P^2 + Pt + n}{Q} \right) \right] \\ &= Qx^2 - (t + 2P)xy + \left(\frac{P^2 + Pt + n}{Q} \right) y^2. \end{aligned}$$

The discriminant of F_γ is

$$\begin{aligned} \Delta &= [-(t + 2P)]^2 - 4Q \left(\frac{P^2 + Pt + n}{Q} \right) \\ &= t^2 + 4tP + 4P^2 - 4P^2 - 4Pt - 4n \\ &= t^2 - 4n. \end{aligned}$$

Hence one associates with γ an indefinite quadratic form F_γ defined as above. The opposite of F_γ is hence

$$\bar{F}_\gamma(x, y) = Qx^2 + (t + 2P)xy + \left(\frac{n + Pt + P^2}{Q} \right) y^2. \tag{11}$$

II. QUADRATICS.

In [6], we derived some results concerning the quadratic irrationals γ , quadratic ideals I_γ and indefinite quadratic forms F_γ defined in (7), (8) and (10), respectively. In the present paper we consider the same problem for other values of Q and P .

Let $\delta = \sqrt{D}$ and $Q = 1$. Then $t = 0$ and $n = -D$. Set $P = \frac{-p}{2}$ for primes p such that $p \equiv 1, 5 \pmod{6}$. Then

$$\gamma_1 = -\frac{p}{2} + \sqrt{D}$$

is a quadratic irrational and hence

$$I_{\gamma_1} = \left[1, \frac{-p}{2} + \sqrt{D} \right] \tag{12}$$

is a quadratic ideal and

$$F_{\gamma_1}(x, y) = x^2 + pxy + \left(\frac{p^2 - 4D}{4} \right) y^2 \tag{13}$$

is a quadratic form of discriminant $\Delta = 4D$.

Theorem 2.1: γ_1 is equivalent to its conjugate $\bar{\gamma}_1$ for every primes $p \equiv 1, 5 \pmod{6}$.

Proof: Recall that two real numbers α and β are said to be equivalent if there exists a $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \bar{\Gamma}$ such that

$$g\alpha = \beta \Leftrightarrow \frac{r\alpha + s}{t\alpha + u} = \beta.$$

The conjugate of γ_1 is $\bar{\gamma}_1 = \frac{-p}{2} - \sqrt{D}$. Now consider the equation

$$g\bar{\gamma}_1 = \gamma_1 \Leftrightarrow \frac{r\left(\frac{-p}{2} - \sqrt{D}\right) + s}{t\left(\frac{-p}{2} - \sqrt{D}\right) + u} = \frac{\frac{-p}{2} + \sqrt{D}}{1} \quad (14)$$

for $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \bar{\Gamma}$. One solution of (14) is

$$g = \begin{pmatrix} -1 & -p \\ 0 & 1 \end{pmatrix} \in \bar{\Gamma}.$$

So γ_1 is equivalent to its conjugate $\bar{\gamma}_1$. ■

Theorem 2.2: I_{γ_1} is ambiguous for every $p \equiv 1, 5 \pmod{6}$.

Proof: We know that an ideal I_γ is ambiguous if it is equal to its conjugate \bar{I}_γ , or in other words, if and only if $\frac{\delta+P}{Q} + \frac{\bar{\delta}+P}{Q} = \frac{t+2P}{Q} \in \mathbf{Z}$. For $\delta = \sqrt{D}$ we have $t = 0$, and hence

$$\frac{t+2P}{Q} = \frac{2(-p/2)}{1} = -p \in \mathbf{Z}. \quad (15)$$

Therefore I_{γ_1} is ambiguous. ■

From above two theorems we can give the following corollary.

Corollary 2.3: F_{γ_1} is properly equivalent to its opposite \bar{F}_{γ_1} and is ambiguous for every $p \equiv 1, 5 \pmod{6}$.

Proof: It is clear that F_{γ_1} is properly equivalent to its opposite \bar{F}_{γ_1} by (15) since $\frac{t+2P}{Q} = -p \in \mathbf{Z}$. We know as above that an indefinite quadratic form F_γ is ambiguous if and only if the quadratic irrational γ is equivalent to its conjugate $\bar{\gamma}$. We proved in Theorem 2.1 that γ_1 is equivalent to its conjugate $\bar{\gamma}_1$. So F_{γ_1} is ambiguous for every $p \equiv 1, 5 \pmod{6}$. ■

Now we can give the following theorem.

Theorem 2.4: Let F_{γ_1} be the quadratic form in (13). Then

- 1) If $p \equiv 1 \pmod{6}$, say $p = 1 + 6k$ for a positive integer $k \geq 1$, then F_{γ_1} is reduced if and only if $D \in [9k^2 + 3k + 1, 9k^2 + 9k + 2] - \{9k^2 + 6k + 1\}$.
- 2) If $p \equiv 5 \pmod{6}$, say $p = 5 + 6k$ for a positive integer $k \geq 1$, then F_{γ_1} is reduced if and only if $D \in [9k^2 + 15k + 7, 9k^2 + 21k + 12] - \{9k^2 + 12k + 9\}$.

In both cases the number of these reduced forms is p .

Proof: 1) Let $p \equiv 1 \pmod{6}$, say $p = 1 + 6k$ and let F_{γ_1} be reduced. Then by (5), we get

$$\begin{aligned} \left| \sqrt{\Delta} - 2|a| \right| < b < \sqrt{\Delta} &\Leftrightarrow \left| \sqrt{4D} - 2|1| \right| < p < \sqrt{4D} \\ &\Leftrightarrow 2\sqrt{D} - 2 < p < 2\sqrt{D}. \end{aligned} \quad (16)$$

Applying (16), we find that

$$D > \frac{p^2}{4} = \frac{1}{4} + 3k + 9k^2 \Leftrightarrow D \geq 9k^2 + 3k + 1$$

and

$$D < \frac{(p+2)^2}{4} = \frac{9}{4} + 9k + 9k^2 \Leftrightarrow D \leq 9k^2 + 9k + 2.$$

So we have

$$9k^2 + 3k + 1 \leq D \leq 9k^2 + 9k + 2.$$

But $D = 9k^2 + 6k + 1 = (3k + 1)^2$ is a square. So we have to omit it (since D must be a square-free positive integer). Therefore we have

$$D \in [9k^2 + 3k + 1, 9k^2 + 9k + 2] - \{9k^2 + 6k + 1\}.$$

The converse is also true, that is, if $D \in [9k^2 + 3k + 1, 9k^2 + 9k + 2] - \{9k^2 + 6k + 1\}$, then F_{γ_1} is reduced. Further the number of these reduced forms is

$$9k^2 + 9k + 2 - (9k^2 + 3k + 1) = 6k + 1 = p.$$

2) Let $p \equiv 5 \pmod{6}$, say $p = 5 + 6k$ and let F_{γ_1} be reduced. Then by (16), we get

$$D > \frac{p^2}{4} = \frac{25}{4} + 15k + 9k^2 \Leftrightarrow D \geq 9k^2 + 15k + 7$$

and

$$D < \frac{(p+2)^2}{4} = \frac{49}{4} + 21k + 9k^2 \Leftrightarrow D \leq 9k^2 + 21k + 12.$$

So we have

$$9k^2 + 15k + 7 \leq D \leq 9k^2 + 21k + 12.$$

But $D = 9k^2 + 18k + 9 = (3k + 3)^2$ is a square. So we have to omit it. Therefore we have

$$D \in [9k^2 + 15k + 7, 9k^2 + 21k + 12] - \{9k^2 + 18k + 9\}.$$

Conversely if $D \in [9k^2 + 15k + 7, 9k^2 + 21k + 12] - \{9k^2 + 18k + 9\}$, then clearly F_{γ_1} is reduced. The number of these reduced forms is

$$9k^2 + 21k + 12 - (9k^2 + 15k + 7) = 6k + 5 = p. \quad \blacksquare$$

Now let $\delta = \frac{1+\sqrt{D}}{2}$ and $Q = 1$. Then $t = 1$ and $n = \frac{1-D}{4}$. Set $P = \frac{-(p+1)}{2}$ for primes p such that $p \equiv 1, 5 \pmod{6}$. Then

$$\gamma_2 = \frac{-p + \sqrt{D}}{2}$$

is a quadratic irrational and hence

$$I_{\gamma_2} = \left[1, \frac{-p + \sqrt{D}}{2} \right] \quad (17)$$

is a quadratic ideal and

$$F_{\gamma_2}(x, y) = x^2 + pxy + \left(\frac{p^2 - D}{4} \right) y^2 \quad (18)$$

is a quadratic form of discriminant $\Delta = D$.

Theorem 2.5: γ_2 is equivalent to its conjugate $\bar{\gamma}_2$ for every $p \equiv 1, 5 \pmod{6}$.

Proof: The conjugate of γ_2 is $\bar{\gamma}_2 = \frac{-p - \sqrt{D}}{2}$. Now consider the equation

$$g\bar{\gamma}_2 = \gamma_2 \Leftrightarrow \frac{r(-p - \sqrt{D}) + s}{t(-p - \sqrt{D}) + u} = \frac{-p + \sqrt{D}}{1} \quad (19)$$

for $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \bar{\Gamma}$. One solution of (19) is

$$g = \begin{pmatrix} -1 & -p \\ 0 & 1 \end{pmatrix} \in \bar{\Gamma}.$$

So γ_2 is equivalent to its conjugate $\bar{\gamma}_2$. ■

Theorem 2.6: I_{γ_2} is ambiguous for every $p \equiv 1, 5 \pmod{6}$.

Proof: Recall that $t = 1$ for $\delta = \frac{1 + \sqrt{D}}{2}$. So

$$\frac{t + 2P}{Q} = \frac{1 + 2\left(\frac{-(p+1)}{2}\right)}{1} = -p \in \mathbf{Z}.$$

Therefore I_{γ_2} is ambiguous. ■

From above two theorems we can give the following corollary.

Corollary 2.7: F_{γ_2} is properly equivalent to its opposite \bar{F}_{γ_2} and is ambiguous for every $p \equiv 1, 5 \pmod{6}$.

Proof: We know that an indefinite quadratic form F_{γ} is ambiguous if and only if the quadratic irrational γ is equivalent to its conjugate $\bar{\gamma}$. We proved in Theorem 2.5 that γ_2 is equivalent to its conjugate $\bar{\gamma}_2$. So F_{γ_2} is ambiguous for every $p \equiv 1, 5 \pmod{6}$. ■

Now we can give the following theorem.

Theorem 2.8: Let F_{γ_2} be the quadratic form in (18). Then

- 1) If $p \equiv 1 \pmod{6}$, say $p = 1 + 6k$ for a positive integer $k \geq 1$, then F_{γ_2} is reduced if and only if $D \in [36k^2 + 12k + 2, 36k^2 + 36k + 8] - \{36k^2 + 24k + 4\}$.
- 2) If $p \equiv 5 \pmod{6}$, say $p = 5 + 6k$ for a positive integer $k \geq 1$, then F_{γ_2} is reduced if and only if $D \in [36k^2 + 60k + 26, 36k^2 + 84k + 48] - \{36k^2 + 72k + 36\}$.

In both cases the number of these reduced forms is $4p + 2$.

Proof: 1) Let $p \equiv 1 \pmod{6}$, say $p = 1 + 6k$ and let F_{γ_2} be reduced. Then by (5), we get

$$\begin{aligned} \left| \sqrt{\Delta} - 2|a| \right| < b < \sqrt{\Delta} &\Leftrightarrow \left| \sqrt{D} - 2|1| \right| < p < \sqrt{D} \\ &\Leftrightarrow \sqrt{D} - 2 < p < \sqrt{D}. \end{aligned} \quad (20)$$

Applying (20) we get

$$D > p^2 = 1 + 12k + 36k^2 \Leftrightarrow D \geq 36k^2 + 12k + 2$$

and

$$D < (p + 2)^2 = 9 + 36k + 36k^2 \Leftrightarrow D \leq 36k^2 + 32k + 8.$$

So

$$36k^2 + 12k + 2 \leq D \leq 36k^2 + 36k + 8.$$

But $D = 36k^2 + 24k + 4 = (6k + 2)^2$ is a square. So we have to omit it. Therefore we have

$$D \in [36k^2 + 12k + 2, 36k^2 + 36k + 8] - \{36k^2 + 24k + 4\}.$$

Conversely if $D \in [36k^2 + 12k + 2, 36k^2 + 36k + 8] - \{36k^2 + 24k + 4\}$, then F_{γ_2} is reduced. Further the number of these reduced forms is

$$36k^2 + 36k + 8 - (36k^2 + 12k + 2) = 24k + 6 = 4p + 2.$$

2) Let $p \equiv 5 \pmod{6}$, say $p = 5 + 6k$ and let F_{γ_2} be reduced. Then by (20), we get

$$D > p^2 = 25 + 60k + 36k^2 \Leftrightarrow D \geq 36k^2 + 60k + 26$$

and

$$D < (p + 2)^2 = 49 + 84k + 36k^2 \Leftrightarrow D \leq 36k^2 + 84k + 48.$$

So we have

$$36k^2 + 60k + 26 \leq D \leq 36k^2 + 84k + 48.$$

But $D = 36k^2 + 72k + 36 = (6k + 6)^2$ is a square. So we have to omit it. Therefore we have

$$D \in [36k^2 + 60k + 26, 36k^2 + 84k + 48] - \{36k^2 + 72k + 36\}.$$

The converse is also true, that is, if $D \in [36k^2 + 60k + 26, 36k^2 + 84k + 48] - \{36k^2 + 72k + 36\}$, then F_{γ_2} is reduced. The number of these reduced forms is

$$36k^2 + 84k + 48 - (36k^2 + 60k + 26) = 24k + 22 = 4p + 2. \quad \blacksquare$$

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