

On the Determination of a Time-like Dual Curve in Dual Lorentzian Space

Emin Özyılmaz

Abstract—In this work, position vector of a time-like dual curve according to standard frame of D_1^3 is investigated. First, it is proven that position vector of a time-like dual curve satisfies a dual vector differential equation of fourth order. The general solution of this dual vector differential equation has not yet been found. Due to this, in terms of special solutions, position vectors of some special time-like dual curves with respect to standard frame of D_1^3 are presented.

Keywords—Classical Differential Geometry, Dual Numbers, Dual Frenet Equations, Time-like Dual Curve, Position Vector, Dual Lorentzian Space.

I. INTRODUCTION

IN the the classical differential geometry, it is well-known that determining position vector of an arbitrary curve according to standard frame is not easy. This problem is important. Because, we think of curves as a geometric set of points, or locus. Intuitively, we are thinking of a curve as the path traced out by a particle moving in the space. Due to this problem, one can see a vast literature on the determining position vectors of the curves according to Frenet frame.

Recently, a method has been developed by B. Y. Chen to classify curves with the solution of differential equations with constant coefficients with respect to standard frame of the space. This method generally uses ordinary vector differential equations as well as Frenet equations. By this way, curves of a finite Chen type and some of classifications are given by the researchers in Euclidean space or another spaces, see [2], [3], [7].

Similar to above method, in [14], the author uses Frenet equations to obtain position vector of a space-like curve according to standard frame of Minkowski space. He constructs a vector differential equation of fourth order and solves it within a special case, by this way, he expresses position vectors of all space-like W-curves in the space E_1^3 .

In recent works, we can see some of papers investigating position vectors of the curves in E^3 or other geometries just like, Minkowski space E_1^3 or dual Lorentzian space D_1^3 . For instance [1], [5], [6], [8], [9], [10]. The aim of these works is to investigate position vectors of the curves with respect to standard Frenet or dual Frenet frame.

In an analogous way as in existing literature, we extend the notion of determine curves by vector differential equations to the time-like dual curves of D_1^3 .

In this work, we use vector differential equations established by means of dual Frenet equations in dual Lorentzian space

Emin Özyılmaz is with the Department of Mathematics, Faculty of Science, Ege University, Bornova-Izmir, Turkey. E-mail: eminozyilmaz@hotmail.com

D_1^3 . We form mentioned vector differential equations to determine position vector of a time-like dual curve according to standard frame of D_1^3 .

II. PRELIMINARIES

To meet the requirements in the next sections, here, the basic elements of the theory of dual curves in the space D_1^3 are briefly presented (A more complete elementary treatment can be found in [1], [6], [8].)

W.K. Clifford, in [4], introduced dual numbers with the set

$$D = \{\hat{x} = x + \xi x^* : x, x^* \in R\}.$$

The symbol ξ designates the dual unit with the property $\xi^2 = 0$ for $\xi \neq 0$. Thereafter, A good amount of reserach work has been done on dual numbers, dual functions and as well as dual curves [1], [6], [9]. Then, dual angle is introduced, which is defined as $\hat{\theta} = \theta + \xi\theta^*$, where θ is the projected angle between two spears and θ^* is the shortest distance between them. The set D of dual numbers is a commutative ring with the operations $(+)$ and (\cdot) . The set

$$D^3 = D \times D \times D = \{\hat{\varphi} : \hat{\varphi} = \varphi + \xi\varphi^*, \varphi \in E^3, \varphi^* \in E^3\}$$

is a module over the ring D , [8].

Let us denote $\hat{a} = a + \xi a^* = (a_1, a_2, a_3) + \xi(a_1^*, a_2^*, a_3^*)$ and $\hat{b} = b + \xi b^* = (b_1, b_2, b_3) + \xi(b_1^*, b_2^*, b_3^*)$. The Lorentzian inner product of \hat{a} and \hat{b} defined by

$$\langle \hat{a}, \hat{b} \rangle = \langle a, b \rangle + \xi(\langle a, b^* \rangle + \langle a^*, b \rangle).$$

We call the dual space D^3 together with Lorentzian inner product as dual Lorentzian space and show by D_1^3 . We call the elements of D_1^3 the dual vectors. For $\hat{\varphi} \neq 0$, the norm $\|\hat{\varphi}\|$ of $\hat{\varphi}$ is defined by $\|\hat{\varphi}\| = \sqrt{|\langle \hat{\varphi}, \hat{\varphi} \rangle|}$. A dual vector $\hat{\varepsilon} = \varepsilon + \xi\varepsilon^*$ is called dual space-like vector if $\langle \hat{\varepsilon}, \hat{\varepsilon} \rangle > 0$ or $\hat{\varepsilon} = 0$, dual time-like vector if $\langle \hat{\varepsilon}, \hat{\varepsilon} \rangle < 0$ and dual null (light-like) vector if $\langle \hat{\varepsilon}, \hat{\varepsilon} \rangle = 0$ for $\hat{\varepsilon} \neq 0$. Therefore, an arbitrary dual curve, which is a differentiable mapping onto D_1^3 , can locally be dual space-like, dual time-like or dual null, if its velocity vector is respectively, dual space-like, dual time-like or dual null. Besides, for the dual vectors $\hat{a}, \hat{b} \in D_1^3$, Lorentzian vector product of dual vectors is defined by

$$\hat{a} \times \hat{b} = a \times b + \xi(a^* \times b + a \times b^*),$$

where $a \times b$ is the classical Lorentzian cross product according to signature $(+, +, -)$, (cf. [11]).

Let $\hat{\varphi} : I \subset R \rightarrow D^3$ be a C^4 time-like dual curve with the arc length parameter s . Then the unit tangent vector is

defined $\hat{\varphi}' = \hat{t}$, and the principal normal is $\hat{n} = \frac{\hat{t}'}{\hat{\kappa}}$, where $\hat{\kappa}$ is never pure-dual. The function $\hat{\kappa} = \|\hat{t}'\| = \kappa + \xi\kappa^*$ is called dual curvature of the dual curve $\hat{\varphi}$. Then the binormal of $\hat{\varphi}$ is given by the dual vector $\hat{b} = \hat{t} \times \hat{n}$. Hence, the triple $\{\hat{t}, \hat{n}, \hat{b}\}$ is called dual Frenet trihedra at the point $\hat{\varphi}$, and dual Frenet formulas may be expressed as [15]

$$\begin{bmatrix} \hat{t}' \\ \hat{n}' \\ \hat{b}' \end{bmatrix} = \begin{bmatrix} 0 & \hat{\kappa} & 0 \\ \hat{\kappa} & 0 & \hat{\tau} \\ 0 & -\hat{\tau} & 0 \end{bmatrix} \begin{bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{bmatrix}, \quad (1)$$

where $\hat{\tau} = \tau + \xi\tau^*$ is the dual torsion of the time-like dual curve $\hat{\varphi}$. Here, we suppose, as the curvature, $\hat{\tau}$ is never pure-dual.

III. POSITION VECTOR OF A TIME-LIKE DUAL SPACE CURVE IN D_1^3

Theorem 1. Let $\hat{\varphi}$ be a time-like dual curve in D_1^3 with nonvanishing curvature and torsion. Position vector $\hat{\varphi}$ satisfies a vector differential equation of fourth order as follows:

$$\frac{d}{ds} \left\{ \frac{1}{\hat{\tau}} \frac{d}{ds} \left(\frac{1}{\hat{\kappa}} \frac{d^2 \hat{\varphi}}{ds^2} \right) + \frac{\hat{\kappa}}{\hat{\tau}} \frac{d\hat{\varphi}}{ds} \right\} - \frac{\hat{\tau}}{\hat{\kappa}} \frac{d\hat{\varphi}}{ds} = 0. \quad (2)$$

Proof: Let $\hat{\varphi}$ be a time-like dual curve of D_1^3 . Then, the system of ordinary vector differential equations (1) hold. Using (1)₁, we write

$$\hat{n} = \frac{1}{\hat{\kappa}} \frac{d\hat{t}}{ds} \quad (3)$$

Substituting (3) to (1)₂, we have

$$\hat{b} = -\frac{1}{\hat{\tau}} \frac{d}{ds} \left(\frac{1}{\hat{\kappa}} \frac{d\hat{t}}{ds} \right) - \frac{\hat{\kappa}}{\hat{\tau}} \hat{t}. \quad (4)$$

And, finally, using (4) in (1)₃, we get

$$\frac{d}{ds} \left\{ \frac{1}{\hat{\tau}} \frac{d}{ds} \left(\frac{1}{\hat{\kappa}} \frac{d\hat{t}}{ds} \right) + \frac{\hat{\kappa}}{\hat{\tau}} \hat{t} \right\} - \frac{\hat{\tau}}{\hat{\kappa}} \hat{t} = 0. \quad (5)$$

Denoting $\frac{d\hat{\varphi}}{ds} = \hat{t}$, we have a vector differential equation fourth order (2) as desired. ■

Proposition 3. The vector differential equation of fourth order (2) is a characterization for the time-like curve $\hat{\varphi} = \hat{\varphi}(s)$. By means of its solution, position vector of an arbitrary time-like dual curve according to standard frame can be determined. Similarly, it is safe to report that if one of dual time-like curves's dual curvature and dual torsion are determined, then position vector of the dual time-like curve can be easily obtained by (2).

However, a general solution of (2) has not yet been found. Therefore, we shall investigate some special cases. Let us first suppose $\hat{\varphi} = \hat{\varphi}(s)$ lies fully in $\hat{t}\hat{n}$ subspace.

Case I: Since $\hat{\tau} \equiv 0$ ($\tau = 0$ and $\tau^* = 0$). Then, we may express

$$\frac{d}{ds} \left(\frac{1}{\hat{\kappa}} \frac{d\hat{t}}{ds} \right) - \hat{\kappa} \hat{t} = 0. \quad (6)$$

Using an exchange variable $\zeta = \int_0^s \hat{\kappa} ds$ in (6), we have

$$\frac{d^2 \hat{t}}{d\zeta^2} - \hat{t} = 0 \quad (7)$$

or in parametric form

$$\begin{cases} \frac{d^2 \hat{t}_1}{d\zeta^2} - \hat{t}_1 = 0 \\ \frac{d^2 \hat{t}_2}{d\zeta^2} - \hat{t}_2 = 0 \\ \frac{d^2 \hat{t}_3}{d\zeta^2} - \hat{t}_3 = 0 \end{cases}, \quad (8)$$

where $\hat{t} = (\hat{t}_1, \hat{t}_2, \hat{t}_3)$ is dual tangent vector according to standard frame of D_1^3 . Using hyperbolic functions, we have the solution

$$\hat{t}_i = \sigma_i \cosh \zeta + \sigma_{i+3} \sinh \zeta. \quad (9)$$

Rewriting the exchange variable, we may express position vector of a dual time-like curve which lies fully in $\hat{t}\hat{n}$ subspace

$$\hat{\varphi} = \begin{pmatrix} \int_0^s \left\{ \sigma_1 \cosh \int_0^s \hat{\kappa} ds + \sigma_4 \sinh \int_0^s \hat{\kappa} ds \right\} ds, \\ \int_0^s \left\{ \sigma_2 \cosh \int_0^s \hat{\kappa} ds + \sigma_5 \sinh \int_0^s \hat{\kappa} ds \right\} ds, \\ \int_0^s \left\{ \sigma_3 \cosh \int_0^s \hat{\kappa} ds + \sigma_6 \sinh \int_0^s \hat{\kappa} ds \right\} ds \end{pmatrix}. \quad (10)$$

Remark 4. It is easy to see that position vector of a dual time-like curve which lies fully in $\hat{t}\hat{n}$ subspace satisfies a vector differential equation of third order.

Case II: Let us suppose curvature and torsion functions of the dual time-like curve $\hat{\varphi} = \hat{\varphi}(s)$ are constant and not congruent to zero ($\kappa = \text{constant}$ and non-zero, $\kappa^* = \text{constant}$ and non-zero; and $\tau = \text{constant}$ and non-zero, $\tau^* = \text{constant}$ and non-zero, i.e.). Then, the equation (2) has the form

$$\frac{d^4 \hat{\varphi}}{ds^4} + (\hat{\kappa}^2 - \hat{\tau}^2) \frac{d^2 \hat{\varphi}}{ds^2} = 0. \quad (11)$$

Solution of (11) depends of character of $\hat{\kappa}^2 - \hat{\tau}^2$. Since, we study the following subcases.

Case II.a: $\kappa = \tau$ and $\kappa^* = \tau^*$. In this case, equation (11) has the form

$$\frac{d^4 \hat{\varphi}}{ds^4} = 0 \quad (12)$$

or in parametric form

$$\left(\frac{d^4 \hat{\varphi}_1}{ds^4}, \frac{d^4 \hat{\varphi}_2}{ds^4}, \frac{d^4 \hat{\varphi}_3}{ds^4} \right) = 0, \quad (13)$$

where $\hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3)$ is the position vector according to standard frame of D_1^3 . Equation (13) yields the position vector

$$\hat{\varphi} = \hat{\varphi}(s) = \begin{pmatrix} \hat{\alpha}_1 \frac{s^3}{6} + \hat{\alpha}_2 \frac{s^2}{2} + \hat{\alpha}_3 s + \hat{\alpha}_4, \\ \hat{\beta}_1 \frac{s^3}{6} + \hat{\beta}_2 \frac{s^2}{2} + \hat{\beta}_3 s + \hat{\beta}_4, \\ \hat{\gamma}_1 \frac{s^3}{6} + \hat{\gamma}_2 \frac{s^2}{2} + \hat{\gamma}_3 s + \hat{\gamma}_4 \end{pmatrix}, \quad (14)$$

where $\hat{\alpha}_i, \hat{\beta}_i, \hat{\gamma}_i \in D$ for $1 \leq i \leq 4$.

Case II.b: $\frac{\kappa}{\kappa^*} = \frac{\tau}{\tau^*}$ and $\kappa > \tau$. Then, we have the solution of (11)

$$\hat{\varphi} = \begin{pmatrix} \hat{\Phi}_1 \cos \sqrt{\hat{\kappa}^2 - \hat{\tau}^2} s + \hat{\Phi}_2 \sin \sqrt{\hat{\kappa}^2 - \hat{\tau}^2} s, \\ \hat{\Theta}_1 \cos \sqrt{\hat{\kappa}^2 - \hat{\tau}^2} s + \hat{\Theta}_2 \sin \sqrt{\hat{\kappa}^2 - \hat{\tau}^2} s, \\ \hat{\Psi}_1 \cos \sqrt{\hat{\kappa}^2 - \hat{\tau}^2} s + \hat{\Psi}_2 \sin \sqrt{\hat{\kappa}^2 - \hat{\tau}^2} s \end{pmatrix}, \quad (15)$$

where $\hat{\Phi}_i, \hat{\Theta}_i, \hat{\Psi}_i \in D$ for $1 \leq i \leq 2$.

Case II.c: $\frac{\kappa}{\kappa^*} = \frac{\tau}{\tau^*}$ and $\kappa < \tau$. In this case, we express the position vector

$$\hat{\varphi} = \begin{pmatrix} \hat{\delta}_1 \cosh \sqrt{\hat{\tau}^2 - \hat{\kappa}^2} s + \hat{\delta}_2 \sinh \sqrt{\hat{\tau}^2 - \hat{\kappa}^2} s, \\ \hat{\omega}_1 \cosh \sqrt{\hat{\tau}^2 - \hat{\kappa}^2} s + \hat{\omega}_2 \sinh \sqrt{\hat{\tau}^2 - \hat{\kappa}^2} s, \\ \hat{\phi}_1 \cosh \sqrt{\hat{\tau}^2 - \hat{\kappa}^2} s + \hat{\phi}_2 \sinh \sqrt{\hat{\tau}^2 - \hat{\kappa}^2} s \end{pmatrix}, \quad (16)$$

where $\hat{\delta}_i, \hat{\omega}_i, \hat{\phi}_i \in D$ for $1 \leq i \leq 2$.

IV. CONCLUSION AND FURTHER REMARKS

In this work, we show that an arbitrary time-like dual curve can be determined by the solution of vector differential equation established by dual Frenet formulas. The presented method opens a door to obtain position vector of the time-like dual curve according standard frame of dual Lorentzian space. However, the general solution of mentioned vector differential equation have not been found. Due to this, we could not generalize position vectors of time-like dual curves. But, by the method of E. Picard successive approximation (cf. [8]), some approximate solutions can be formed. By this way, behaviour of the curve (moving particle, i.e.) may be addressed.

In recent years, dual numbers have been applied to study the motion of a line in space; they seem even to be most appropriate way for this end and they have triggered use of dual numbers in kinematical problems. For instance [11], [12], [13], [15]. We hope these results will be helpful to the mathematicians who are specialized on mathematical modelling.

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