The Slant Helices According to Bishop Frame

Bahaddin Bukcu, Murat Kemal Karacan

Abstract**—**In this study, we have defined slant helix according to Bishop frame in Euclidean 3-Space. Furthermore, we have given some necassary and sufficient conditons for the slant helix.

Keywords**—**Slant helix, Bishop frame, Parallel transport frame

I. INTRODUCTION

LET M be an n-dimensional smooth manifold equipped with a metric $\langle , \rangle$..A tangent space $T_p(M)$ at a point $p \in M$ is furnished with the canonical inner product. If $\langle , \rangle$ is positive definite, then M is a Riemannian manifold. A curve on an Riemannian Manifold M is a smooth mapping $\alpha : I \rightarrow M$, where I is an open interval in the real line R^1. As an open submanifold of R^1, I has a coordinate system consisting of the identity map u of I. The velocity vector of α at $s \in I$

$$\alpha'(s) = \frac{d\alpha(u)}{du}\Big|_s \in T_{\alpha(s)}M.$$

A curve $\alpha(s)$ is said to be regular if $\alpha'(s)$ is not equal to zero for any s. Let $\alpha(s)$ be a curve on M ,denote by $\{T, N, B\}$ the moving Frenet frame along the curve α .Then T, N and B are the tangent, the principal normal and binormal vectors of the curve α respectively. If α is a space curve , then this set of orthogonal unit vectors, known as the Frenet-Serret frame, has the following properties

$$\begin{aligned} &\alpha'(s) = T \\ &D_T T = \kappa N \\ &D_T N = -\kappa T + \tau B \\ &D_T T = -\tau N, \end{aligned}$$

where D denotes the covariant differentiation in M .
In a Riemann manifold M , a curve is described by the Frenet formula. For example, if all curvatures of a curve are identically zero, then the curve a geodesic. If only the curvature κ is a non-zero constant and the torsion τ is all identically zero, then the curve is called a circle. If the curvature κ and the torsion τ are non-zero constants, then the curve is called a helix. If the curvature κ and the torsion τ are not constant but $\frac{\kappa}{\tau}$ is constant, then the curve is a called a general helix [4,7].

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. we can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that, while $T(s)$ for a given curve model is unique, we may choose any convenient arbitrary basis $(N_1(s), N_2(s))$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $T(s)$ at each point.If the derivatives of $(N_1(s), N_2(s))$ depend only on $T(s)$ and not each other we can make $N_1(s)$ and $N_2(s)$ vary smoothly throughout the path regardless of the curvature. Therefore, we have the alternative frame equations

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}. \qquad (1)$$

One can show (see, Bishop [3]) that

$$\begin{aligned} &\kappa(s) = \sqrt{k_1^2 + k_2^2} \\ &\theta(s) = \arctan\left(\frac{k_2}{k_1}\right), k_1 \neq 0 \\ &\tau(s) = -\frac{d\theta(s)}{ds} \end{aligned}$$

so that k_1 and k_2 effectively correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = -\int \tau(s)ds\,\theta$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ (and hence from the Frenet frame) due to the differentiation [1,2].

Bahaddin Bukcu is with the Gazi Osmanpasa University, Faculty of Sciences and Arts,Department of Mathematics,Tokat-Turkey (e-mail: bbukcu@yahoo.com).
Murat Kemal Karacan is with Usak University, Faculty of Sciences and Arts, Department of Mathematics 64200, Usak-Turkey (corresponding author ; e-mail: murat.karacan@usak.edu.tr).

II. THE SLANT HELICES ACCORDING TO BISHOP FRAME

Definition 2.1. A regular curve $\alpha : I \to E^3$ is called a slant helix provided the unit vector $N_1(s)$ of α has constant angle θ with some fixed unit vector u; that is, $\langle N_1(s), u\rangle = \cos\theta$ for all $s \in I$.

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. Slant helices can be identified by a simple condition on natural curvatures.

Theorem 2.1. Let $\alpha : I \to E^3$ be a unit speed curve with nonzero natural curvatures. Then α is a slant helix if and only if $\dfrac{k_1}{k_2}$ is constant.

Proof. Let α be slant helix in E^3 and $\langle N_1, u\rangle = consnt.$ Then α is slant helix; from the definition, we have

$$\langle N_1, u\rangle = consnt.$$

where u is a unit vector, called the axis of slant helix. By differentiation we get

$$\langle N_1', u\rangle = \langle -k_1 T, u\rangle = -k_1\langle T, u\rangle = 0.$$

Hence

$$\langle T, u\rangle = 0.$$

Again differentiating from the last equality, we can write as follows

$$\begin{aligned}\langle T', u\rangle &= \langle k_1 N_1 + k_2 N_2, u\rangle \\ &= k_1\langle N_1, u\rangle + k_2\langle N_2, u\rangle \\ &= k_1\cos\theta + k_2\sin\theta = 0.\end{aligned}$$

Therefore we obtain

$$\frac{k_1}{k_2} = -\tan\theta$$

as desired.

Suppose that $\dfrac{k_1}{k_2} = -\tan\theta$. Then we can write $u \in Sp\{N_1, N_2\}$, i.e.,

$$u = N_1\cos\theta + N_2\sin\theta.$$

Differentiating the last equality,

$$u' = (k_1\cos\theta + k_2\sin\theta)T = 0.$$

So u is a constant vector. Thus, the proof is done.

Theorem 2.2. Let $\alpha = \alpha(s)$ be a unit speed curve in E^3. Then α is a slant helix iff

$$\det(N_1', N_1'', N_1''') = 0.$$

Proof. $(\Rightarrow:)$ Suppose that $\dfrac{k_1}{k_2}$ be constant. We have equalities as

$$\begin{aligned}-N_1' &= k_1 T \\ -N_1'' &= k_1{}' T + k_1^2 N_1 + k_1 k_2 N_2 \\ -N_1''' &= \left(k_1{}'' - k_1^3 - k_1 k_2^2\right)T \\ &\quad + \left(3k_1 k_1{}'\right)N_1 + \left(2k_1{}' k_2 + k_1 k_2{}'\right)N_2\end{aligned}$$

So we get

$$\det(N_1',' N_1'', N_1''') = k_1^2\begin{bmatrix}1 & 0 & 0 \\ * & k_1 & k_2 \\ \circ & 3k_1k_1' & 2k_1'k_2 + k_1k_2'\end{bmatrix}$$

$$= k_1\left(\frac{k_1}{k_2}\right)^2\left(\frac{k_1}{k_2}\right)'.$$

Since α is a slant helix, $\dfrac{k_1}{k_2}$ is constant. Hence, we have

$$\det(N_1', N_1'', N_1''') = 0, k_2 \neq 0.$$

$(\Leftarrow:)$ Suppose that $\det(N_1', N_1'', N_1''') = 0$. Then it is clear that the $\dfrac{k_1}{k_2} = const.$ for being

$$\left(\frac{k_1}{k_2}\right)' = 0.$$

Theorem 2.3. Let $\alpha = \alpha(s)$ be a unit speed curve in E^3. Then α is a slant helix iff

$$\det(N_2', N_2'', N_2''') = 0$$

Proof. $(\Rightarrow:)$ Suppose that $\dfrac{k_1}{k_2}$ be constant. From Eq. (1) one can find

$$-N_2' = -k_2 T$$

and

$$\begin{aligned}-N_2'' &= (k_2')T + (k_1k_2)N_1 + (k_1k_2)N_2, \\ -N_2''' &= \left(k_2'' - k_1^2k_2 - k_2^3\right)T \\ &\quad + (2k_1k_2' + k_1'k_2)N_1 + (3k_2k_2')N_2.\end{aligned}$$

Moreover, we have

$$\det\left(N_2',' N_2'', N_2'''\right) = -k_2^2 \begin{bmatrix} 1 & 0 & 0 \\ * & k_1 & k_2 \\ \circ & 2k_1k_2' + k_1'k_2 & 3k_2k_2' \end{bmatrix}$$

$$= k_2^5 \left(\frac{k_1}{k_2}\right)'.$$

Since α is a slant helix curve $\frac{k_1}{k_2}$ is constant. Hence, we have

$$\det\left(N_2', N_2'', N_2'''\right) = 0$$

$(\Leftarrow:)$ Suppose that $\det\left(N_2', N_2'', N_2'''\right) = 0$. Then it is clear that the $\frac{k_1}{k_2} = const.$ for being

$$\left(\frac{k_1}{k_2}\right)' = 0.$$

Next we consider general slant helices in a Euclidean manifold M. Then we have equalities

$$\alpha'(s) = T$$
$$D_T T = k_1 N_1 + k_2 N_2$$
$$D_T N_1 = -k_1 T$$
$$D_T N_2 = -k_2 T,$$

for any $s \in I$, where $N_1(s)$ and $N_2(s)$ are vector fields and k_1 and k_2 are functions of parameter s.

Theorem 2.4. A unit speed curve α on M is a general slant helix iff

$$D_T\left(D_T D_T N_1\right) = -A D_T N_1 - 3k_1' D_T T \qquad (2)$$

where

$$A = \kappa^2 - \frac{k_1''}{k_1}, k_1^2 + k_2^2 = \kappa^2. \qquad (3)$$

Proof. Suppose that α is general slant helix. Then, from Eq. (2), we have

$$\begin{aligned} D_T\left(D_T N_1\right) &= D_T\left(-k_1 T\right) = -k_1' T - k_1 D_T T \\ &= -k_1' T - k_1^2 N_1 - k_1 k_2 N_2 \end{aligned} \qquad (4)$$

and

$$\begin{aligned} D_T\left(D_T D_T N_1\right) &= \left(-k_1'' + k_1 k_2^2\right) T - k_1^2 D_T N_1 \\ &\quad - 2k_1 k_1' N_1 - 3k_1' D_T T \\ &\quad - \left(k_1' k_2 - k_1 k_2'\right) N_2 - k_1' D_T T. \end{aligned} \qquad (5)$$

Now, since α is a general slant helix, we have

$$\frac{k_1}{k_2} = const.$$

and this upon the derivation give rise to

$$k_1' k_2 = k_1 k_2'.$$

If we substitute the values

$$T = -\frac{1}{k_1} D_T N_1 \qquad (6)$$

and

$$\left(k_1 k_2\right)' = 2k_1' k_2,$$

in Eq.(2.4) we obtain

$$D_T\left(D_T D_T N_1\right) = \left(\frac{k_1''}{k_1} - \kappa^2\right) D_T N_1 - 3k_1' D_T T.$$

$$D_T\left(D_T D_T N_1\right) = \left(\frac{k_1''}{k_1} - \kappa^2\right) D_T N_1 - 3k_1' D_T T.$$

So we get as desired.

Conversely let us assume that Eq. (2) holds. We show that the curve α is general slant helix. Differentiating covariantly Eq. (6) we obtain

$$\begin{aligned} D_T T &= D_T\left(-\frac{1}{k_1} D_T N_1\right) \\ &= \frac{k_1'}{k_1^2} D_T N_1 - \frac{1}{k_1} D_T D_T N_1 \end{aligned}$$

and so,

$$\begin{aligned} D_T D_T T &= \left(\frac{k_1'}{k_1^2}\right)' D_T N_1 + \frac{k_1'}{k_1^2} D_T D_T N_1 \\ &\quad + \frac{k_1'}{k_1^2} D_T D_T N_1 - \frac{1}{k_1} D_T D_T D_T N_1 \end{aligned} \qquad (7)$$

If we use Eq. (2) in Eq. (7), we get

$$D_T D_T T = \left[\left(\frac{k_1'}{k_1^2}\right)' + \frac{A}{k_1}\right] D_T N_1 + \frac{2k_1'}{k_1^2} D_T D_T N_1$$
$$+ \frac{3k_1'}{k_1} D_T T_1$$

Substituting Eq. (4) and Eq. (5) in this last equaility we obtain

$$D_T D_T T = \left[\left(\frac{k_1'}{k_1^2}\right)' + \frac{A}{k_1}\right] D_T N_1 - \frac{2k_1^2 k_1'}{k_1^2} T -$$
$$- 2k_1' N_1 - \frac{2k_1' k_2}{k_1} N_2 + 3k_1' N_1 + \frac{3k_1' k_2}{k_1} N_2.$$

From the last equality we have

$$D_T D_T T = \left[\left(\frac{k_1'}{k_1^2} \right)' + \frac{A}{k_1} \right] D_T N_1 - \frac{2k_1'^2}{k_1^2} T + k_1' N_1 + \frac{k_1' k_2}{k_1} N_2 . \tag{8}$$

On the other hand we can write $D_T D_T T$ as follows

$$D_T D_T T = k_1 D_T N_1 - k_2^2 T + k_1' N_1 + k_2' N_2 . \tag{9}$$

From comparision the Eq. (8) and Eq. (9) we obtain equialities below

$$\frac{k_1' k_2}{k_1} = k_2'$$

and so

$$\frac{k_1'}{k_1} = \frac{k_2'}{k_2} . \tag{10}$$

Integrating Eq. (10), we get

$$\frac{k_1}{k_2} = const.$$

Thus α is a general slant helix. Hence, the proof is done.

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