

4-Transitivity and 6-Figures in Finite Klingenberg Planes of Parameters (p^{2k-1}, p)

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Abstract—In this paper, we carry over some of the results which are valid on a certain class of Moufang-Klingenberg planes $\mathbf{M}(\mathcal{A})$ coordinatized by an local alternative ring $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$ of dual numbers to finite projective Klingenberg plane $\mathbf{M}(\mathcal{A})$ obtained by taking local ring \mathbf{Z}_q (where prime power $q = p^k$) instead of \mathbf{A} . So, we show that the collineation group of $\mathbf{M}(\mathcal{A})$ acts transitively on 4-gons, and that any 6-figure corresponds to only one invertible $m \in \mathcal{A}$.

Keywords—finite Klingenberg plane, projective collineation, 4-transitivity, 6-figures.

I. INTRODUCTION

Projective Klingenberg and Hjelmslev planes (more briefly: PK-planes and PH-planes, resp.) are generalizations of ordinary projective planes. These structures were introduced by Klingenberg in [14], [15]. As for finite PK-planes, these structures introduced by Drake and Lenz in [8] have been studied in detail by Bacon in [2].

In our previous paper [6] we studied a certain class (which we will denote by $\mathbf{M}(\mathcal{A})$) of Moufang-Klingenberg (briefly, MK) planes coordinatized by an local alternative ring

$$\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$$

of dual numbers (an alternative ring \mathbf{A} , $\varepsilon \notin \mathbf{A}$ and $\varepsilon^2 = 0$) introduced by Blunck in [5]. We showed that its collineation group is transitive on quadrangles and the coordinatization of these Moufang-Klingenberg planes is independent of the choice of the coordinatization quadrangle. By extending the concepts of 6-figure to these Moufang - Klingenberg planes, we examined some properties of 6-figures.

In the present paper we deal with finite PK-plane $\mathbf{M}(\mathcal{A})$ obtained by taking local ring \mathbf{Z}_q (where q is a prime power) instead of \mathbf{A} . So, we will carry the results that are well-known for MK-planes from [6] $\mathbf{M}(\mathcal{A})$ to the finite PK-plane $\mathbf{M}(\mathcal{A})$.

II. PRELIMINARIES

Let $\mathbf{M} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$ (points, lines, incidence) and an equivalence relation ' \sim ' (neighbour relation) on \mathbf{P} and on \mathbf{L} . Then \mathbf{M} is called a *projective Klingenberg plane* (PK-plane), if it satisfies the following axioms:

(PK1) If P, Q are two non-neighbour points, then there is a unique line PQ through P and Q .

(PK2) If g, h are two non-neighbour lines, then there is a unique point $g \wedge h$ on both g and h .

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(PK3) There is a projective plane $\mathbf{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$ and incidence structure epimorphism $\Psi : \mathbf{M} \rightarrow \mathbf{M}^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q, \Psi(g) = \Psi(h) \Leftrightarrow g \sim h$$

hold for all $P, Q \in \mathbf{P}$, $g, h \in \mathbf{L}$.

PK-plane \mathbf{M} is called a *projective Hjelmslev plane* (PH-plane) if \mathbf{M} furthermore provides the following axioms:

(PH1) If P, Q are two neighbour points, then there are at least two lines through P and Q .

(PH2) If g, h are two neighbour lines, then there are at least two points on both g and h .

A *Moufang-Klingenberg plane* (MK-plane) is a PK-plane \mathbf{M} that generalizes a Moufang plane, and for which \mathbf{M}^* is a Moufang plane (for the details see [1]).

A point $P \in \mathbf{P}$ is called *near* a line $g \in \mathbf{L}$ iff there exists a line h such that $P \in h$ for some line $h \sim g$.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a *collineation* of \mathbf{M} .

Now we give the definition of an n -gon, which is meaningful when $n \geq 3$: An n -tuple of pairwise non-neighbour points is called an (ordered) *n -gon* if no three of its elements are on neighbour lines [6].

An *alternative ring (field)* \mathbf{R} is a not necessarily associative ring (field) that satisfies the alternative laws $a(ab) = a^2b$, $(ba)a = ba^2$, $\forall a, b \in \mathbf{R}$. An alternative ring \mathbf{R} with identity element 1 is called *local* if the set \mathbf{I} of its non-unit elements is an ideal.

We summarize some basic concepts about the coordinatization of MK-planes from [3].

Let \mathbf{R} be a local alternative ring. Then

$$\mathbf{M}(\mathbf{R}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$$

is the incidence structure with neighbor relation defined as follows:

$$\begin{aligned} \mathbf{P} &= \{(x, y, 1) : x, y \in \mathbf{R}\} \cup \{(1, y, z) : y \in \mathbf{R}, z \in \mathbf{I}\} \\ &\quad \cup \{(w, 1, z) : w, z \in \mathbf{I}\} \\ \mathbf{L} &= \{[m, 1, p] : m, p \in \mathbf{R}\} \cup \{[1, n, p] : p \in \mathbf{R}, n \in \mathbf{I}\} \\ &\quad \cup \{[q, n, 1] : q, n \in \mathbf{I}\} \end{aligned}$$

$$\begin{aligned}
 [m, 1, p] &= \{(x, xm + p, 1) : x \in \mathbf{R}\} \\
 &\cup \{(1, zp + m, z) : z \in \mathbf{I}\} \\
 [1, n, p] &= \{(yn + p, y, 1) : y \in \mathbf{R}\} \\
 &\cup \{(zp + n, 1, z) : z \in \mathbf{I}\} \\
 [q, n, 1] &= \{(1, y, yn + q) : y \in \mathbf{R}\} \\
 &\cup \{(w, 1, wq + n) : w \in \mathbf{I}\}
 \end{aligned}$$

and also

$$\begin{aligned}
 P &= (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = Q \\
 \Leftrightarrow &x_i - y_i \in \mathbf{I} \ (i = 1, 2, 3), \forall P, Q \in \mathbf{P}
 \end{aligned}$$

$$\begin{aligned}
 g &= [x_1, x_2, x_3] \sim [y_1, y_2, y_3] = h \\
 \Leftrightarrow &x_i - y_i \in \mathbf{I} \ (i = 1, 2, 3), \forall g, h \in \mathbf{L}.
 \end{aligned}$$

Baker *et al.* [1] use $(O = (0, 0, 1), U = (1, 0, 0), V = (0, 1, 0), E = (1, 1, 1))$ as a coordinatization 4-gon. We stick to this notation throughout this paper. For more detailed information about the coordinatization see [1] and [3].

Now it is time to give the following theorem from [1].

Theorem 2.1: $\mathbf{M}(\mathbf{R})$ is an MK-plane, and each MK-plane is isomorphic to some $\mathbf{M}(\mathbf{R})$.

Let \mathbf{A} be an alternative field and $\varepsilon \notin \mathbf{A}$. Consider $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$ with componentwise addition and multiplication as follows:

$$(a_1 + a_2\varepsilon)(b_1 + b_2\varepsilon) = a_1b_1 + (a_1b_2 + a_2b_1)\varepsilon,$$

where $a_i, b_i \in \mathbf{A}$, $i = 1, 2$. Then \mathcal{A} is an alternative ring with ideal $\mathbf{I} = \mathbf{A}\varepsilon$ of non-units. For more detailed information about \mathcal{A} see the papers of [4], [5].

Theorem 2.2: If \mathbf{R} is a (not necessarily commutative) local ring then $\mathbf{M}(\mathbf{R})$ is a PK-plane (cf. [15] or [9, Theorem 4.1]).

Drake and Lenz [8, Proposition 2.5] or [12, Theorem 1.2] observed that the following corollary is true for PK-planes. This corollary is a generalization of results which are given for PH-planes by Kleinfeld [13, Theorem 1] and Lüneburg [16, Satz 2.11].

Corollary 2.3: Let $\mathbf{M}(\mathbf{R})$ be PK-plane. Then there are natural numbers t and r which are called the parametres of $\mathbf{M}(\mathbf{R})$ and they are uniquely determined by incidence structure of a finite PK-plane [8, Proposition 2.7], with

- 1) every point (line) has t^2 neighbours;
- 2) given a point P and a line l with $P \in l$, there exist exactly t points on l which are neighbours to P and exactly t lines through P which are neighbours to l ;
- 3) Let r be order of the projective plane \mathbf{M}^* . If $t \neq 1$ we have $r \leq t$ (then \mathbf{M} is called *proper*; we have $t = 1$ iff \mathbf{M} is an ordinary projective plane)
- 4) every point (line) is incident with $t(r + 1)$ lines (points);
- 5) $|\mathbf{P}| = |\mathbf{L}| = t^2(r^2 + r + 1)$.

Now consider ring \mathbf{Z}_q where prime power $q = p^k$. We can state the elements of \mathbf{Z}_q as $\mathbf{Z}_q = U' \cup I$ where U' is the set of units of \mathbf{Z}_q and I is the set of non-units of \mathbf{Z}_q . Here it is clear that

$$I = \{0p, 1p, 2p, \dots, (p^{k-1} - 1)p\}$$

and so $|I| = p^{k-1}$. Let $\varepsilon \notin \mathbf{Z}_q$. Then $\mathcal{A} := \mathbf{Z}_q + \mathbf{Z}_q\varepsilon$ with componentwise addition and multiplication above is a local ring with ideal $\mathbf{I} := I + \mathbf{Z}_q\varepsilon$ of non-units, $|\mathbf{I}| = (p^{k-1})p^k$. Note that the set of units of \mathcal{A} is $\mathbf{U} := U' + \mathbf{Z}_q\varepsilon$ and

$$|\mathbf{U}| = (p^k - p^{k-1})p^k = (p - 1)p^{2k-1}.$$

Since \mathcal{A} is a proper local ring and $\mathcal{A}/\mathbf{I} = \mathbf{Z}_p$, Ψ induces an incidence structure epimorphism from finite PK-plane $\mathbf{M}(\mathcal{A})$ onto the Desarguesian projective plane (with order p) coordinatized by the field \mathbf{Z}_p [9, page 169, above Theorem 4.1]. Because of this, $\mathbf{M}(\mathcal{A})$ is called as Desarguesian PK-plane.

So, we have the following

Corollary 2.4: For finite PK-plane $\mathbf{M}(\mathcal{A})$, the parameters t and r in Corollary 2.3 are equal to p^{2k-1} and p , respectively.

A local ring \mathbf{R} is called a *Hjelmslev ring* (briefly, H-ring) if it satisfies the following two conditions:

(HR1) \mathbf{I} consists of two-sided zero divisor.

(HR2) For $a, b \in \mathbf{I}$, one has $a \in b\mathbf{R}$ or $b \in a\mathbf{R}$, and also $a \in \mathbf{R}b$ or $b \in \mathbf{R}a$.

By the last definition, we can say that \mathcal{A} is not a H-ring. For example, for elements $a = 3 + 3\varepsilon$ and $b = \varepsilon$ of the ideal \mathbf{I} of local ring $\mathcal{A} = \mathbf{Z}_{3^2} + \mathbf{Z}_{3^2}\varepsilon$, (HR2) is not valid.

From now on we restrict ourselves to PK-plane $\mathbf{M}(\mathcal{A}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$ coordinatized by the local ring $\mathcal{A} := \mathbf{Z}_q + \mathbf{Z}_q\varepsilon$, with neighbour relation defined above.

III. 4-TRANSITIVITY AND 6-FIGURES IN $\mathbf{M}(\mathcal{A})$.

In the final section, first of all, from [6] we start by giving some collineations on $\mathbf{M}(\mathcal{A})$ where $w, z, q, n \in \mathbf{I}$ as follows:

For any $a, b \in \mathcal{A}$, the collineation $T_{a,b}$ transforms points and lines as follows:

$$\begin{aligned}
 (x, y, 1) &\rightarrow (x + a, y + b, 1) \\
 (1, y, z) &\rightarrow (1, y + z(b - ay), z) \\
 (w, 1, z) &\rightarrow (w + za, 1, z)
 \end{aligned}$$

and

$$\begin{aligned}
 [m, 1, k] &\rightarrow [m, 1, k + b - am] \\
 [1, n, p] &\rightarrow [1, n, p + a - bn] \\
 [q, n, 1] &\rightarrow [q, n, 1].
 \end{aligned}$$

For any $\alpha, \beta \notin \mathbf{I}$, the collineation $S_{\alpha, \beta}$ (here, it is enough to give $S_{\alpha, \beta}$ instead of the collineations L_a and F_a in [6]) transforms points and lines as follows:

$$\begin{aligned}
 (x, y, 1) &\rightarrow (\beta x, \alpha y, 1) \\
 (1, y, z) &\rightarrow (1, \alpha\beta^{-1}y, \beta^{-1}z) \\
 (w, 1, z) &\rightarrow (\alpha^{-1}\beta w, 1, \alpha^{-1}z)
 \end{aligned}$$

and

$$\begin{aligned} [m, 1, k] &\rightarrow [\alpha\beta^{-1}m, 1, \alpha k] \\ [1, n, p] &\rightarrow [1, \alpha^{-1}\beta n, \beta p] \\ [q, n, 1] &\rightarrow [\beta^{-1}q, \alpha^{-1}n, 1]. \end{aligned}$$

The collineation I_1 transforms points and lines as follows:

$$\begin{aligned} (x, y, 1) &\rightarrow (x^{-1}, x^{-1}y, 1) \quad \text{if } x \notin \mathbf{I} \\ (x, y, 1) &\rightarrow (1, y, x) \quad \text{if } x \in \mathbf{I} \\ (1, y, z) &\rightarrow (z, y, 1) \\ (w, 1, z) &\rightarrow (z, 1, w) \end{aligned}$$

and

$$\begin{aligned} [m, 1, k] &\rightarrow [k, 1, m] \\ [1, n, p] &\rightarrow [p, n, 1] \quad \text{if } p \in \mathbf{I} \\ [1, n, p] &\rightarrow [1, -np^{-1}, p^{-1}] \quad \text{if } p \notin \mathbf{I} \\ [q, n, 1] &\rightarrow [1, n, q]. \end{aligned}$$

The collineation F transforms points and lines as follows:

$$\begin{aligned} (x, y, 1) &\rightarrow (y, x, 1) \\ (1, y, z) &\rightarrow (y, 1, z) \quad \text{if } y \in \mathbf{I} \\ (1, y, z) &\rightarrow (1, y^{-1}, y^{-1}z) \quad \text{if } y \notin \mathbf{I} \\ (w, 1, z) &\rightarrow (1, w, z) \end{aligned}$$

and

$$\begin{aligned} [m, 1, k] &\rightarrow [1, m, k] \quad \text{if } m \in \mathbf{I} \\ [m, 1, k] &\rightarrow [m^{-1}, 1, -km^{-1}] \quad \text{if } m \notin \mathbf{I} \\ [1, n, p] &\rightarrow [n, 1, p] \\ [q, n, 1] &\rightarrow [n, q, 1]. \end{aligned}$$

For any $s \in \mathcal{A}$, the collineation G_s transforms points and lines as follows:

$$\begin{aligned} (x, y, 1) &\rightarrow (x, y - xs, 1) \\ (1, y, z) &\rightarrow (1, y - s, z) \\ (w, 1, z) &\rightarrow (w, 1, z) \end{aligned}$$

and

$$\begin{aligned} [m, 1, k] &\rightarrow [m - s, 1, k] \\ [1, n, p] &\rightarrow [1, n, p + psn] \\ [q, n, 1] &\rightarrow [q + sn, n, 1]. \end{aligned}$$

The collineation I_2 transforms points and lines as follows:

$$\begin{aligned} (x, y, 1) &\rightarrow (y^{-1}x, y^{-1}, 1) \quad \text{if } y \notin \mathbf{I} \\ (x, y, 1) &\rightarrow (1, x^{-1}, x^{-1}y) \quad \text{if } y \in \mathbf{I} \wedge x \notin \mathbf{I} \\ (x, y, 1) &\rightarrow (x, 1, y) \quad \text{if } y \in \mathbf{I} \wedge x \in \mathbf{I} \\ (1, y, z) &\rightarrow (y^{-1}, y^{-1}z, 1) \quad \text{if } y \notin \mathbf{I} \\ (1, y, z) &\rightarrow (1, z, y) \quad \text{if } y \in \mathbf{I} \\ (w, 1, z) &\rightarrow (w, z, 1) \end{aligned}$$

and

$$\begin{aligned} [m, 1, k] &\rightarrow [-mk^{-1}, 1, k^{-1}] \quad \text{if } k \notin \mathbf{I} \\ [m, 1, k] &\rightarrow [1, -km^{-1}, m^{-1}] \quad \text{if } k \in \mathbf{I} \wedge m \notin \mathbf{I} \\ [m, 1, k] &\rightarrow [m, k, 1] \quad \text{if } k \in \mathbf{I} \wedge m \in \mathbf{I} \\ [1, n, p] &\rightarrow [p^{-1}, 1, -np^{-1}] \quad \text{if } p \notin \mathbf{I} \\ [1, n, p] &\rightarrow [1, p, n] \quad \text{if } p \in \mathbf{I} \\ [q, n, 1] &\rightarrow [q, 1, n]. \end{aligned}$$

So, we can give the following theorem without proof. For, its proof is same to Theorem 2 of [6]. Furthermore, this theorem is proved by Lemma 4.15 in [11].

Theorem 3.1: The group \mathcal{G} of collineations of $\mathbf{M}(\mathcal{A})$ acts transitively on 3-gons.

Now, we can state the analogue of the result given by [2, Proposition 5.2.10 in Vol.I]. For the case of uniform H-rings (for the definition of uniform see [10]), the result is also in [7, Theorem 17]. Here, it is possible to give the proof of the following theorem, as more shortly than the proof of Theorem 3 in [6].

Theorem 3.2: \mathcal{G} acts transitively on 4-gons of $\mathbf{M}(\mathcal{A})$.

Proof: Let (P, Q, R, S) be a 4-gon in $\mathbf{M}(\mathcal{A})$. It suffices to show that the points P, Q, R, S can be transformed by an element of \mathcal{G} to $U, V, (1, 1, 1), O$, respectively. From Theorem 3.1, there exists a collineation σ which transforms P, Q, R to $U, V, (0, 1, 1)$, respectively. Let E denote the intersection point of the lines QR and PS . Then, since $\sigma(E)$ is non-neighbour to the points $\sigma(P), \sigma(Q), \sigma(R)$, it has the form $(0, b, 1)$, where $b - 1 \notin \mathbf{I}$, and so $\sigma(S)$ has the form $(a, b, 1)$, where $a \notin \mathbf{I}$. Therefore σ transforms P, Q, R, S to

$$(1, 0, 0), (0, 1, 0), (0, 1, 1), (a, b, 1),$$

respectively. Then the mapping $T_{-a, -b}$ transforms these points to

$$(1, 0, 0), (0, 1, 0), (-a, 1 - b, 1), (0, 0, 1),$$

respectively and $S_{(1-b)^{-1}, -a^{-1}}$ transforms these points to

$$(1, 0, 0), (0, 1, 0), (1, 1, 1), (0, 0, 1),$$

respectively. ■

The following corollary is an obvious result of the last theorem:

Corollary 3.3: The coordinatization of $\mathbf{M}(\mathcal{A})$ is independent of the choice of the coordinatization base.

From now on, we carry over some concepts related to 6-figures to the $\mathbf{M}(\mathcal{A})$, in view of the paper of [6].

A 6-figure is a sequence of six non-neighbour points $(ABC, A_1B_1C_1)$ such that (A, B, C) is 3-gon, and $A_1 \in$

$BC, B_1 \in CA, C_1 \in AB$. The points A, B, C, A_1, B_1, C_1 are called vertices of this 6-figure. The 6-figures $(ABC, A_1B_1C_1)$ and $(DEF, D_1E_1F_1)$ are *equivalent* if there exists a collineation of $\mathbf{M}(\mathcal{A})$ which transforms A, B, C, A_1, B_1, C_1 to D, E, F, D_1, E_1, F_1 respectively. Now, we give a theorem from [6].

Theorem 3.4: Let $\mu = (ABC, A_1B_1C_1)$ be a 6-figure in $\mathbf{M}(\mathcal{A})$. Then, there is an $m \in \mathbf{U}$ such that μ is equivalent to $(UVO, (0, 1, 1)(1, 0, 1)(1, m, 0))$ where $U = (1, 0, 0), V = (0, 1, 0), O = (0, 0, 1)$ are elements of the coordinatization basis of $\mathbf{M}(\mathcal{A})$.

We again give a theorem from [6]. Note that the proof of this theorem is more shorter.

Theorem 3.5: The 6-figures

$$(ABC, A_1B_1C_1), (BCA, B_1C_1A_1), (CAB, C_1A_1B_1)$$

are equivalent.

Proof: By Theorem 3.4 we may without loss of generality take $(UVO, U_1V_1O_1)$ instead of $(ABC, A_1B_1C_1)$, where

$$U_1 = (0, 1, 1), V_1 = (1, 0, 1), O_1 = (1, m, 0)$$

with $m \in \mathbf{U}$. The collineation

$$h := S_{m,1} \circ I_2 \circ I_1$$

transforms $(UVO, U_1V_1O_1)$ to $(VOU, V_1O_1U_1)$ and also $(VOU, V_1O_1U_1)$ to $(OUV, O_1U_1V_1)$. ■

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