Vector Sequence Space S_2

K.Chandrasekhara Rao and K.Balasubramanian

Abstract—In this paper we discuss the various properties of the space s_2. The sequences are taken with terms in a commutative Banach algebra X with identity *e* and we have obtained the following results. s_2 is a non separable Banach space which is solid and has monotone norm.

Keywords—Vector sequence spaces, s_2 space, solid space.

I. s_2 SPACE

Definition:- Let *X* be a commutative Banach algebra with identity *e*. s_2 is the set of all sequences (x_k) with $x_k \in X$ and $\frac{\|x_k\|}{2^k} \leq M \forall k$ where $M > 0$ is a constant. Let$\|.\|$ be the norm in *X*. The norm of the sequence is $\|\|.\|\|$.

II. PROPERTIES OF s_2 SPACE

Proposition (1)

$s_2 \subset l_\infty$

$where\, l_\infty = \{all\ bounded\ sequences\ \ (x_k)\, with\ x_k \in X \forall k\}.$

Proof

$We\ have\ l_\infty = \{x : \|x_k\| \leq M\}.$

$$But\ \frac{\|x_k\|}{2^k} \leq \|x_k\| \leq M \forall k$$

$$and\ s_2 = \left\{(x_k) : \sup_{(k)} \frac{\|x_k\|}{2^k} < \infty\right\}.$$

$$Hence\ s_2 \subset l_\infty.$$

Theorem 1

$$s_2\ is\ a\ normed\ space.$$

Proof

$$Put\ \ \|\|x\|\| = \sup_{(k)} \frac{\|x_k\|}{2^k}.$$

$$We\ have\ \frac{\|x_k\|}{2^k} \geq 0 \Rightarrow \|\|x\|\| \geq 0 \qquad (1)$$

$$\frac{\|x_k\|}{2^k} = 0 \Leftrightarrow \|\|x\|\| = 0$$

$$Hence\, \|\|x\|\| = 0 \Leftrightarrow x = 0 \qquad (2)$$

$$\|\|\alpha x\|\| = \sup_{(k)} \frac{\|\alpha x_k\|}{2^k} = |\alpha| \sup_{(k)} \frac{\|x_k\|}{2^k}$$

$$= |\alpha|\ \|\|x\|\|\, where\ \alpha\ is\ a\ scalar \qquad (3)$$

$$\|\|x+y\|\| = \sup_{(k)} \left\{\frac{\|x_k + y_k\|}{2^k}\right\}$$

$$\leq \sup_{(k)} \frac{\|x_k\| + \|y_k\|}{2^k} = \|\|x\|\| + \|\|y\|\| \qquad (4)$$

$$Thus\ \ \|\|x+y\|\| \leq \|\|x\|\| + \|\|y\|\|.$$

$From\ (1),\ (2),\ (3)\ and\ (4),\ \|\|x\|\|\ is\ the\ norm\ of\ x.$

$$\Rightarrow s_2\ is\ a\ normed\ space.$$

Theorem 2

$$s_2\ is\ a\ Banach\ space.$$

Proof

$Let\ \left\{x^{(n)}\right\}_{n=1}^{\infty} be\ a\ Cauchy\ sequence\ in\ s_2\, where$

$$x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, ...\right) \forall n.$$

$In\ other\ words,$

$$x^{(1)} = \left(x_1^{(1)}, x_2^{(1)}, ...\right)$$

$$x^{(2)} = \left(x_1^{(2)}, x_2^{(2)}, ...\right)$$

$$..........$$

$$..........$$

$$\Rightarrow \left\|\left\|x^{(n)} - x^{(m)}\right\|\right\| \leq \epsilon \forall n, m \geq n_0.$$

$$\Rightarrow \frac{\left\|x_k^{(n)} - x_k^{(m)}\right\|}{2^k} < \epsilon \forall n, m \geq n_0.$$

$$\Rightarrow \left\|x_k^{(n)} - x_k^{(m)}\right\| \leq \epsilon 2^k \forall n, m \geq n_0.$$

$$\Rightarrow \left\{x_k^{(n)}\right\}_{n=1}^{\infty}\ is\ a\ Cauchy\ sequence\ in\ X.$$

$But\ X\ is\ complete.$

$Hence\ x_k^{(n)} \rightarrow x_k\ as\ n \rightarrow \infty.$

$$Take\ x = \left\{\frac{x_k}{2^k}\right\}$$

$.\ Then\ x \in s_2\ and\ x^{(n)} \rightarrow x\ in\ s_2.$

$$Therefore\ s_2\ is\ complete.$$

Theorem 3

$$s_2\ is\ not\ seperable.$$

Proof

$Let\ D\ be\ any\ dense\ subset\ of\ l_\infty.$

$Let\ A\ be\ the\ set\ of\ all\ these\ sequences\ whose\ terms\ are\ 0\ or\ 1.$

K.Chandrasekhara Rao, Department of Mathematics, Srinivasa Ramanujan Centre, SASTRA University, Kumbakonam-612001,INDIA. e-mail : k.chandrasekhararao@hotmail.com

K.Balasubramanian, Department of Mathematics, SASTRA University, Thanjavur-613402, INDIA. e-mail :*k_bala27@yahoo.co.in*

$Then\ A\ is\ an\ uncountable\ subset\ of\ l_\infty.$
$Define\ a\ surjection\ f : A \to D\ by$

$$f(x) = z_x \forall x = (x_1, x_2, ..., x_n, ...) \in A\ with$$

$$|||x - z_x||| < \frac{1}{2} \tag{5}$$

$Let\ x, y \in A\ with\ x \neq y.$

$$But\ then\ |||x - y||| = \sup_{(k)} \left\| \frac{x_k}{2^k} - \frac{x_k}{2^k} \right\| = 1 \tag{6}$$

$$Now\ |||x - y||| \leq |||x - z_x||| + |||z_x - y|||$$

$$and\ so\ |||y - z_x||| \geq |||x - y||| - |||x - z_x||| \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

$$But\ f(y) = z_y\ with\ |||y - z_y||| < \frac{1}{2}.$$

$$Hence\ z_x \neq z_y\ or\ equivalently,\ f(x) \neq f(y).$$

$$Thus,\ f\ is\ a\ bijection.$$

$Since\ A\ is\ uncountable\ it\ follows\ that$

$$f(A) = D\ is\ uncountable.$$

$$Consequently,\ s_2\ cannot\ be\ separable.$$

Theorem 4

$$s_2\ is\ solid.$$

Proof

$$s_2 = \left\{ (x_k) : \frac{\|x_k\|}{2^k} \leq M for\ some\ M > 0 \right\}.$$

$Suppose\ that\ (x_k) \in s_2.$

$$Let\ \|u_k\| \leq \|x_k\| \forall k \tag{7}$$

$$Hence\ \frac{\|u_k\|}{2^k} \leq \frac{\|x_k\|}{2^k} \forall k \tag{8}$$

$$But\ (x_k) \in s_2.$$

$$\Rightarrow \frac{\|x_k\|}{2^k} \leq M for\ some\ M > 0 \tag{9}$$

$From\ (8)\ and\ (9)\ we\ get$

$$\frac{\|u_k\|}{2^k} \leq \frac{\|x_k\|}{2^k} \leq M.$$

$$\Rightarrow \frac{\|u_k\|}{2^k} \leq M$$

$$\Rightarrow (u_k) \in s_2$$

$$Therefore\ s_2\ is\ solid.$$

Theorem 5

$$s_2^\alpha = s_2^\beta = s_2^\gamma = T_2\ where\ T_2 = \left\{ (y_k) : \sum_{k=1}^{\infty} 2^k \|y_k\| < \infty \right\}.$$

Proof

$$Since\ s_2\ is\ solid,\ \alpha, \beta, \gamma\ -\ duals\ are\ equal.$$

$$We\ shall\ show\ that\ s_2^\beta = T_2.$$

$$Let\ y = (y_k) \in T_2.$$

$$Then\ \left\| \sum_{k=1}^{\infty} x_k y_k \right\| \leq \sum_{k=1}^{\infty} \|x_k\| \|y_k\|$$

$$= \sum_{k=1}^{\infty} \frac{\|x_k\|}{2^k} 2^k \|y_k\|$$

$$\leq |||x||| \sum_{k=1}^{\infty} 2^k \|y_k\|$$

$$because\ |||x||| = \sup_{(k)} \frac{\|x_k\|}{2^k}.$$

$$Hence\ y \in s_2^\beta.$$

$$Therefore\ T_2 \subset s_2^\beta.\ Similarly\ s_2^\beta \subset T_2.$$

$$Therefore\ \beta\ -\ dual\ of\ s_2\ is\ T_2.$$

$$Thus\ s_2^\alpha = s_2^\beta = s_2^\gamma = T_2.$$

Theorem 6

$$s_2\ has\ monotone\ norm.$$

Proof

$Let\ m > n.\ Consider\ the\ sequence$

$$x^{[n]} = (x_1, x_2, ...x_n, 0, 0, ...).$$

$$But\ then\ x^{[m]} = (x_1, x_2, ...x_n, ..., x_m, 0, 0, ...).$$

$$Hence\ \left|\left|\left| x^{[n]} \right|\right|\right| = \sup \left\{ \frac{\|x_1\|}{2}, \frac{\|x_2\|}{2^2}, ..., \frac{\|x_n\|}{2^n}, 0, 0, ... \right\}$$

$$\left|\left|\left| x^{[m]} \right|\right|\right| = \sup \left\{ \frac{\|x_1\|}{2}, \frac{\|x_2\|}{2^2}, ..., \frac{\|x_n\|}{2^n}, ..., \frac{\|x_m\|}{2^m}, 0, 0, ... \right\}$$

$Obviously,\ for\ m > n,\ we\ have$

$$\left|\left|\left| x^{[m]} \right|\right|\right| \geq \left|\left|\left| x^{[n]} \right|\right|\right| \tag{10}$$

$$Also\ \lim_{n \to \infty} \left|\left|\left| x^{[n]} \right|\right|\right| = |||x|||$$

$$\Rightarrow \inf_{(n)} \left|\left|\left| x^{[n]} \right|\right|\right| = |||x||| \tag{11}$$

$$From\ (10)\ and\ (11),\ s_2\ has\ monotone\ norm.$$

Definition

$$c_{02} = \left\{ (x_k) \in s_2 : \frac{\|x_k\|}{2^k} \to 0\ as\ k \to \infty \right\}$$

Theorem 7

$$c_{02}\ has\ AK\ property.$$

Proof

$$Let\ x = (x_k) \in c_{02}.$$

$$Then\ x^{[n]} = (x_1, x_2, ...x_n, 0, 0, ...).$$

$$\Rightarrow x - x^{[n]} = (x_{n+1}, x_{n+2}, ...).$$

$$\Rightarrow \left|\left|\left| x - x^{[n]} \right|\right|\right| = \sup_{k > n} \frac{\|x_k\|}{2^k} \to 0\ as\ k \to \infty,\ because\ (x_k) \in c_{02}.$$

$$Hence\ c_{02}\ has\ AK\ property.$$

III. CONCLUSION

Thus we have proved s_2 as a non separable Banach space which is solid and has monotone norm.

REFERENCES

[1] K.Chandrasekhara Rao, *Functional Analysis*,Alpha Science, Oxford, 2006.
[2] Johann Boos, *Classical and Modern methods in summability*,Oxford University Press, Oxford, 2006.
[3] P.K Kamthan and Manjul Gupta, *Sequence Spaces and Series*,,Marcel Deckker, New York - Basel,1981.
[4] A.Wilansky, *Summability through Functional Analysis*,North - Holland, Amsterdam, 1984.