

# Generalized inverse eigenvalue problems for symmetric arrow-head matrices

Yongxin Yuan

*Abstract*—In this paper, we first give the representation of the general solution of the following inverse eigenvalue problem (IEP): Given  $X \in \mathbf{R}^{n \times p}$  and a diagonal matrix  $\Lambda \in \mathbf{R}^{p \times p}$ , find nontrivial real-valued symmetric arrow-head matrices  $A$  and  $B$  such that  $AX\Lambda = BX$ . We then consider an optimal approximation problem: Given real-valued symmetric arrow-head matrices  $\tilde{A}, \tilde{B} \in \mathbf{R}^{n \times n}$ , find  $(\hat{A}, \hat{B}) \in \mathcal{S}_E$  such that  $\|\hat{A} - \tilde{A}\|^2 + \|\hat{B} - \tilde{B}\|^2 = \min_{(A,B) \in \mathcal{S}_E} (\|A - \tilde{A}\|^2 + \|B - \tilde{B}\|^2)$ , where  $\mathcal{S}_E$  is the solution set of IEP. We show that the optimal approximation solution  $(\hat{A}, \hat{B})$  is unique and derive an explicit formula for it.

*Keywords*—partially prescribed spectral information, symmetric arrow-head matrix, inverse problem, optimal approximation.

## I. INTRODUCTION

**T**HROUGHOUT this paper, we denote the real  $m \times n$  matrix space by  $\mathbf{R}^{m \times n}$ , the set of all symmetric matrices in  $\mathbf{R}^{n \times n}$  by  $\mathbf{SR}^{n \times n}$ , the transpose and the Moore-Penrose generalized inverse of a real matrix  $A$  by  $A^T$  and  $A^+$ , respectively.  $I_n$  represents the identity matrix of size  $n$ . For  $A, B \in \mathbf{R}^{m \times n}$ , an inner product in  $\mathbf{R}^{m \times n}$  is defined by  $(A, B) = \text{trace}(B^T A)$ , then  $\mathbf{R}^{m \times n}$  is a Hilbert space. The matrix norm  $\|\cdot\|$  induced by the inner product is the Frobenius norm. Given two matrices  $A = [a_{ij}] \in \mathbf{R}^{m \times n}$  and  $B = [b_{ij}] \in \mathbf{R}^{p \times q}$ , the Kronecker product of  $A$  and  $B$  is defined by  $A \otimes B = [a_{ij} B] \in \mathbf{R}^{mp \times nq}$ . Also, for an  $m \times n$  matrix  $A = [a_1, a_2, \dots, a_n]$ , where  $a_i, i = 1, \dots, n$ , is the  $i$ -th column vector of  $A$ , the stretching function  $\text{Vec}(A)$  is defined by  $\text{Vec}(A) = [a_1^T, a_2^T, \dots, a_n^T]^T$ .

**Definition 1** An  $n \times n$  matrix  $A$  is called an arrow-head matrix if

$$A = \begin{bmatrix} a_1 & b_1 & b_2 & \cdots & b_{n-1} \\ c_1 & a_2 & 0 & \cdots & 0 \\ c_2 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & 0 & 0 & \cdots & a_n \end{bmatrix}.$$

If  $b_i = c_i, i = 1, \dots, n-1$ , then  $A$  is a symmetric arrow-head matrix.

The application background and the computations of the eigenvalues and eigenvectors of this kind of matrices can see

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[1, 2, 3, 4]. The inverse problem of constructing the symmetric arrow-head matrix from spectral data has been investigated by Peng et al. [5], and Borges et al. [6]. In this paper we will further consider generalized inverse eigenvalue problems for symmetric arrow-head matrices, which can be described as follows:

**Problem IEP.** Given  $X \in \mathbf{R}^{n \times p}$  and a diagonal matrix  $\Lambda \in \mathbf{R}^{p \times p}$ , find nontrivial real-valued symmetric arrow-head matrices  $A$  and  $B$  such that

$$AX\Lambda = BX. \tag{1}$$

**Problem II.** Given real-valued symmetric arrow-head matrices  $\tilde{A}, \tilde{B} \in \mathbf{R}^{n \times n}$ , find  $(\hat{A}, \hat{B}) \in \mathcal{S}_E$  such that

$$\|\hat{A} - \tilde{A}\|^2 + \|\hat{B} - \tilde{B}\|^2 = \min_{(A,B) \in \mathcal{S}_E} (\|A - \tilde{A}\|^2 + \|B - \tilde{B}\|^2), \tag{2}$$

where  $\mathcal{S}_E$  is the solution set of IEP.

The paper is organized as follows. In Section 2, using the Kronecker product and stretching function  $\text{Vec}(\cdot)$  of matrices, we give an explicit representation of the solution set  $\mathcal{S}_E$  of Problem IEP. In Section 3, we show that there exists a unique solution in Problem II and present the expression of the unique solution  $(\hat{A}, \hat{B})$  of Problem II. Finally, in Section 4, a numerical algorithm to acquire the optimal approximation solution under the Frobenius norm sense is described and a numerical example is provided.

## II. THE SOLUTION OF PROBLEM IEP

To begin with, we introduce two lemmas.

**Lemma 1:** [7] If  $L \in \mathbf{R}^{m \times q}, b \in \mathbf{R}^m$ , then  $Ly = b$  has a solution  $y \in \mathbf{R}^q$  if and only if  $LL^+b = b$ . In this case, the general solution of the equation can be described as  $y = L^+b + (I_q - L^+L)z$ , where  $z \in \mathbf{R}^q$  is an arbitrary vector.

**Lemma 2:** [8] Let  $D \in \mathbf{R}^{m \times n}, H \in \mathbf{R}^{n \times l}, J \in \mathbf{R}^{l \times s}$ . Then

$$\text{Vec}(DHJ) = (J^T \otimes D)\text{Vec}(H).$$

Let  $S_0$  be the set of all real-valued symmetric arrow-head matrices, then  $S_0$  is a linear subspace of  $\mathbf{SR}^{n \times n}$ , and the dimension of  $S_0$  is  $d = 2n - 1$ .

Define  $Y_{ij}$  as

$$Y_{ij} = \begin{cases} \frac{\sqrt{2}}{2}(e_i e_j^T + e_j e_i^T), & i = 1, j = 2, \dots, n; \\ e_i e_i^T, & i = j = 1, \dots, n, \end{cases} \tag{3}$$

where  $e_i$ ,  $i = 1, \dots, n$ , is the  $i$ -th column vector of the identity matrix  $I_n$ . It is easy to verify that  $\{Y_{ij}\}$  forms an orthonormal basis of the subspace  $S_0$ , that is,

$$(Y_{ij}, Y_{kl}) = \begin{cases} 0, & i \neq k \text{ or } j \neq l, \\ 1, & i = k \text{ and } j = l. \end{cases} \quad (4)$$

Now, if  $A, B \in \mathbf{R}^{n \times n}$  are symmetric arrow-head matrices, then  $A, B$  can be expressed as

$$A = \sum_{i,j} \alpha_{ij} Y_{ij}, \quad B = \sum_{i,j} \beta_{ij} Y_{ij}, \quad (5)$$

where the real numbers  $\alpha_{ij}, \beta_{ij}$ ,  $\begin{cases} i = 1, j = 2, \dots, n; \\ i = j = 1, \dots, n, \end{cases}$  are yet to be determined. Substituting (5) into (1), we have

$$\sum_{i,j} \alpha_{ij} Y_{ij} X \Lambda - \sum_{i,j} \beta_{ij} Y_{ij} X = 0. \quad (6)$$

Let

$$\begin{aligned} \alpha &= [\alpha_{11}, \dots, \alpha_{n,n}, \alpha_{12}, \dots, \alpha_{1,n}]^T, \\ \beta &= [\beta_{11}, \dots, \beta_{n,n}, \beta_{12}, \dots, \beta_{1,n}]^T, \\ G &= [\text{Vec}(Y_{11}), \dots, \text{Vec}(Y_{n,n}), \\ &\quad \text{Vec}(Y_{12}), \dots, \text{Vec}(Y_{1,n})] \in \mathbf{R}^{n^2 \times d} \end{aligned} \quad (7)$$

and

$$M = (\Lambda^T X^T \otimes I_n)G, \quad N = (X^T \otimes I_n)G. \quad (8)$$

Using Lemma 2, we see that the equation of (6) is equivalent to

$$M\alpha - N\beta = 0. \quad (9)$$

It follows from Lemma 1 that the equation of (9) with unknown vector  $\alpha$  has a solution if and only if

$$E_M N \beta = 0, \quad (10)$$

where  $E_M = I_{np} - MM^+$ . Using Lemma 1 again, we know that the equation of (10) with respect to  $\beta$  is always solvable and the general solution to the equation is

$$\beta = (I_d - (E_M N)^+ E_M N)u, \quad (11)$$

where  $u \in \mathbf{R}^d$  is an arbitrary vector.

Substituting (11) into (9) and applying Lemma 1, we obtain

$$\alpha = M^+ N (I_d - (E_M N)^+ E_M N)u + F_M v, \quad (12)$$

where  $F_M = I_d - M^+ M$ , and  $v \in \mathbf{R}^d$  is an arbitrary vector.

In summary of above discussion, we have proved the following result.

**Theorem 1:** Suppose that  $X \in \mathbf{R}^{n \times p}$ ,  $\Lambda \in \mathbf{R}^{p \times p}$ , and  $\Lambda$  is a diagonal matrix. Let  $\{Y_{ij}\}$ ,  $G, M, N$  be given as in (3), (7) and (8). Write  $d = 2n - 1$ ,  $E_M = I_{np} - MM^+$ ,  $F_M =$

$I_d - M^+ M$ . Then the solution set  $\mathcal{S}_E$  of Problem IEP can be expressed as

$$\mathcal{S}_E = \{(A, B) \in \mathbf{SR}^{n \times n} \times \mathbf{SR}^{n \times n} : A = K(\alpha \otimes I_n), B = K(\beta \otimes I_n)\}, \quad (13)$$

where

$$K = [Y_{11}, \dots, Y_{n,n}, Y_{12}, \dots, Y_{1,n}] \in \mathbf{R}^{n \times nd}, \quad (14)$$

$\alpha, \beta$  are, respectively, given by (12) and (11) with  $u, v \in \mathbf{R}^d$  being arbitrary vectors.

### III. THE SOLUTION OF PROBLEM II

It follows from Theorem 1 that the set  $\mathcal{S}_E$  is always nonempty. It is easy to verify that  $\mathcal{S}_E$  is a closed convex subset of  $\mathbf{SR}^{n \times n} \times \mathbf{SR}^{n \times n}$ . From the best approximation theorem [9], we know there exists a unique solution  $(\hat{A}, \hat{B})$  in  $\mathcal{S}_E$  such that (2) holds.

We now focus our attention on seeking the unique solution  $(\hat{A}, \hat{B})$  in  $\mathcal{S}_E$ . For the real-valued symmetric arrow-head matrices  $\tilde{A}$  and  $\tilde{B}$ , it is easily seen that  $\tilde{A}, \tilde{B}$  can be expressed as the linear combinations of the orthonormal basis  $\{Y_{ij}\}$ , that is,

$$\tilde{A} = \sum_{i,j} \gamma_{ij} Y_{ij}, \quad \tilde{B} = \sum_{i,j} \delta_{ij} Y_{ij}, \quad (15)$$

where  $\gamma_{ij}, \delta_{ij}$ ,  $\begin{cases} i = 1, j = 2, \dots, n; \\ i = j = 1, \dots, n, \end{cases}$  are uniquely determined by the elements of  $\tilde{A}$  and  $\tilde{B}$ . Let

$$\gamma = [\gamma_{11}, \dots, \gamma_{n,n}, \gamma_{12}, \dots, \gamma_{1,n}]^T, \quad (16)$$

$$\delta = [\delta_{11}, \dots, \delta_{1,n}, \delta_{12}, \dots, \delta_{1,n}]^T. \quad (17)$$

Then, for any pair of matrices  $(A, B) \in \mathcal{S}_E$  in (13), by the relations of (4) and (15) we see that

$$\begin{aligned} f &= \|A - \tilde{A}\|^2 + \|B - \tilde{B}\|^2 \\ &= \left\| \sum_{i,j} (\alpha_{ij} - \gamma_{ij}) Y_{ij} \right\|^2 + \left\| \sum_{i,j} (\beta_{ij} - \delta_{ij}) Y_{ij} \right\|^2 \\ &= \left( \sum_{i,j} (\alpha_{ij} - \gamma_{ij}) Y_{ij}, \sum_{i,j} (\alpha_{ij} - \gamma_{ij}) Y_{ij} \right) \\ &\quad + \left( \sum_{i,j} (\beta_{ij} - \delta_{ij}) Y_{ij}, \sum_{i,j} (\beta_{ij} - \delta_{ij}) Y_{ij} \right) \\ &= \sum_{i,j} (\alpha_{ij} - \gamma_{ij}) (Y_{ij}, \sum_{i,j} (\alpha_{ij} - \gamma_{ij}) Y_{ij}) \\ &\quad + \sum_{i,j} (\beta_{ij} - \delta_{ij}) (Y_{ij}, \sum_{i,j} (\beta_{ij} - \delta_{ij}) Y_{ij}) \\ &= \sum_{i,j} (\alpha_{ij} - \gamma_{ij})^2 + \sum_{i,j} (\beta_{ij} - \delta_{ij})^2 \\ &= \|\alpha - \gamma\|^2 + \|\beta - \delta\|^2. \end{aligned}$$

Substituting (11) and (12) into the relation of  $f$ , we have

$$\begin{aligned} f &= \|M^+NWu + F_Mv - \gamma\|^2 + \|Wu - \delta\|^2 \\ &= u^TWN^T(MM^T)^+NWu - 2\gamma^TM^+NWu \\ &\quad - 2\gamma^TF_Mv + v^TF_Mv + \gamma^T\gamma \\ &\quad + u^TWu - 2u^TW\delta + \delta^T\delta, \end{aligned}$$

where  $W = I_d - (E_MN)^+E_MN$ . Therefore,

$$\begin{aligned} \frac{\partial f}{\partial u} &= 2WN^T(MM^T)^+NWu \\ &\quad - 2WN^T(M^+)^T\gamma + 2Wu - 2W\delta, \\ \frac{\partial f}{\partial v} &= 2F_Mv - 2F_M\gamma. \end{aligned}$$

Clearly,  $\|A - \tilde{A}\|^2 + \|B - \tilde{B}\|^2 = \min$  if and only if

$$\frac{\partial f}{\partial u} = 0, \quad \frac{\partial f}{\partial v} = 0$$

which yields

$$\begin{aligned} Wu &= (I_d + WN^T(MM^T)^+NW)^{-1}W(\delta + N^T(M^+)^T\gamma), \\ &\quad (18) \\ F_Mv &= F_M\gamma. \end{aligned} \quad (19)$$

Upon substituting (18) and (19) into (11) and (12), we obtain

$$\hat{\alpha} = M^+NW(I_d + WN^T(MM^T)^+NW)^{-1}W(\delta + N^T(M^+)^T\gamma) + F_M\gamma, \quad (20)$$

$$\hat{\beta} = W(I_d + WN^T(MM^T)^+NW)^{-1}W(\delta + N^T(M^+)^T\gamma). \quad (21)$$

By now, we have proved the following result.

**Theorem 2:** Let the real-valued symmetric arrow-head matrices  $\hat{A}$  and  $\hat{B}$  be given. Then Problem II has a unique solution and the unique solution of Problem II can be expressed as

$$\hat{A} = K(\hat{\alpha} \otimes I_n), \quad (22)$$

$$\hat{B} = K(\hat{\beta} \otimes I_n), \quad (23)$$

where  $\hat{\alpha}$ ,  $\hat{\beta}$  are given by (20) and (21), respectively.

#### IV. A NUMERICAL EXAMPLE

Based on Theorem 1 and Theorem 2 we can describe an algorithm for solving Problem IEP and Problem II as follows.

##### Algorithm 1.

- 1) Input  $\tilde{A}$ ,  $\tilde{B}$ ,  $\Lambda$ ,  $X$ .
- 2) Form the orthonormal basis  $\{Y_{ij}\}$  by (3).
- 3) Compute  $G$ ,  $M$ ,  $N$  according to (7) and (8), respectively.
- 4) Compute  $E_M = I_{np} - MM^+$ ,  $F_M = I_d - M^+M$ ,  $W = I_d - (E_MN)^+E_MN$ .
- 5) Form vectors  $\gamma$ ,  $\delta$  by (15), (16) and (17).
- 6) Compute  $K$ ,  $\hat{\alpha}$ ,  $\hat{\beta}$  by (14), (20) and (21), respectively.
- 7) Compute the unique solution  $(\hat{A}, \hat{B})$  of Problem II by (22) and (23).

##### Example 1. Given

$$\tilde{A} = \begin{pmatrix} -4 & 2 & 5 & 1 & 2 & 11 \\ 2 & -3 & 0 & 0 & 0 & 0 \\ 5 & 0 & -6 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 & -4 & 0 \\ 11 & 0 & 0 & 0 & 0 & -44 \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} -7 & 2 & 19 & 9 & 3 & 15 \\ 2 & -13 & 0 & 0 & 0 & 0 \\ 19 & 0 & -8 & 0 & 0 & 0 \\ 9 & 0 & 0 & -6 & 0 & 0 \\ 3 & 0 & 0 & 0 & -3 & 0 \\ 15 & 0 & 0 & 0 & 0 & -28 \end{pmatrix}$$

and

$$\begin{aligned} \Lambda &= \text{diag} \{\lambda_1, \lambda_2, \lambda_3\} \\ &= \text{diag} \{2.1709, 0.84882, 0.73245\}, \end{aligned}$$

$$\begin{aligned} X &= [x_1, x_2, x_3] \\ &= \begin{bmatrix} -0.27362 & 0.019321 & -0.090308 \\ 0.071468 & 0.00087224 & -0.0056621 \\ 0.70165 & 0.085116 & -0.3485 \\ -0.91 & 0.035512 & -0.16064 \\ -0.050465 & -0.91 & -0.39781 \\ -0.030195 & -0.024079 & 0.91 \end{bmatrix}. \end{aligned}$$

Using Algorithm 1, we obtain the unique solution of Problem II as follows.

$$\hat{A} = \begin{bmatrix} -3.8904 & 1.8001 & 4.7078 \\ 1.8001 & -2.7001 & 0 \\ 4.7078 & 0 & -5.6494 \\ 0.90628 & 0 & 0 \\ 1.8874 & 0 & 0 \\ 10.383 & 0 & 0 \\ 0.90628 & 1.8874 & 10.383 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1.8126 & 0 & 0 \\ 0 & -3.7749 & 0 \\ 0 & 0 & -41.531 \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} -6.6982 & 2.0161 & 20.037 \\ 2.0161 & -13.105 & 0 \\ 20.037 & 0 & -8.4365 \\ 9.1354 & 0 & 0 \\ 3.1709 & 0 & 0 \\ 15.857 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 9.1354 & 3.1709 & 15.857 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -6.0902 & 0 & 0 \\ 0 & -3.1709 & 0 \\ 0 & 0 & -29.6 \end{bmatrix},$$

We define the residual as

$$\text{res}(\lambda_i, x_i) = \|(\lambda_i \hat{A} - \hat{B})x_i\|,$$

and the numerical results shown as follows.

$(\lambda_i, x_i)$	$(\lambda_1, x_1)$	$(\lambda_2, x_2)$	$(\lambda_3, x_3)$
Res $(\lambda_i, x_i)$	1.7468e-014	1.0116e-014	3.6636e-014

Furthermore, we can figure out

$$\|\hat{A} - \tilde{A}\| = 2.7319, \quad \|\hat{B} - \tilde{B}\| = 2.57.$$

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