Some New Subclasses of Nonsingular H-matrices

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Abstract—In this paper, we obtain some new subclasses of nonsingular H-matrices by using α diagonally dominant matrix.

Keywords—H-matrix, diagonal dominance, α diagonally dominant matrix.

I. Introduction

LET $A \quad (a_{ij})_{n\times n} \in C^{n\times n}$, $M(A) \quad (m_{ij})$, where

$$m_{ij} \quad \begin{cases} |a_{ii}|, i \quad j, \\ -|a_{ij}|, i \;/\; j, \end{cases} \quad i \quad 1,2,...,n.$$

Then we call $M(A)$ is the comparison matrix of A. Suppose A is an n by n matrix over the field of real numbers. If A can be expressed in the form $A \quad \sigma I - B$ where B is a nonnegative matrix and $\sigma > \rho(B)$ the spectral radius of B, then A is called a nonsingular M-matrix. This class of matrices has been much studied [1].

If $M(A)$ is nonsingular M-matrix, then A is called a nonsingular H-matrix. If

$$|a_{ii}| > \sum_{j\neq i} |a_{ij}|, \; i \quad 1,2,\cdots,n,$$

then we say A is strictly diagonally dominant. If there exist positive number $x_1, x_2, \cdots x_n$ such that

$$x_i\,|a_{ii}| > \sum_{j\neq i} x_j\,|a_{ij}|, \; i \quad 1,2,\cdots,n \quad ,$$

then we say A is generalized strictly diagonally dominant[2].

A matrix A be a nonsingular H-matrix is equivalent to that A be a generalized strictly diagonally dominant matrix [3].

H-matrices have important applications, for instance, in iterative methods of numerical analysis, in the analysis of dynamical systems, in economics, and in mathematical programming. But how to determine whether an n by n complex matrix is a nonsingular H-matrix is not easy in practice. In this paper, we will give some new subclasses of nonsingular H-matrices.

II. Main Results

We will use the following notations:

$$R_i(\;) \quad \sum_{j\neq i} |a_{ij}|, \;\; S_i(\;) \quad \sum_{j\neq i} |a_{ji}|,$$
$$i \in \langle n\rangle \cdot \quad \{1,2,\cdots,n\},$$

$$I(A) \quad \left\{ \nu \in S(A) \;\middle|\; \begin{array}{l} \prod_{i\in\nu} |a_{ij}| \;/\; \prod_{i\in\nu} R_i(A) \\ or \; \prod_{i\in\nu} |a_{ij}| \;/\; \prod_{i\in\nu} C_i(A) \end{array} \right\},$$

$$^{\alpha}_{2} \quad \{i \in <n> \mid |a_{ij}| > \alpha R_i(A) + (1-\alpha) S_i(A)\},$$
$$^{\alpha}_{1} \quad <n> \setminus \; ^{\alpha}_{2},$$

$$\beta_\alpha \quad \{i \in <n> \mid \alpha R_i(A) + (1-\alpha) S_i(A) > 0\}.$$

Definition [4] Let $A \quad (a_{ij}) \in C^{n\times n}$. If there exists $\alpha \in \; 0,1$, $|a_{ii}| \geq \alpha R_i + (1-\alpha) S_i (i \in N)$ holds, then we call A is α diagonally dominant and denote $A \in D_0(\alpha)$. If all the inequations are strict, we denote $A \in D(\alpha)$.

Lemma 1[4] Let $A \quad (a_{ij}) \in C^{n\times n}$. If $A \in D(\alpha)$, then A is a nonsingular H-matrix.

Lemma 2[4] Let $A \quad (a_{ij}) \in C^{n\times n}$. If for $\alpha \in \; 0,1$, $|a_{ii}| \geq \alpha R_i + (1-\alpha) S_i$ holds, and for i which satisfies $|a_{ii}| \quad \alpha R_i + (1-\alpha) S_i$ there exists a non-zero elements chain $a_{ii_1}, a_{i_1 i_2}, \ldots, a_{i_p j} \;/\; 0$ such that $j \in J \quad \{j \in N \mid |a_{ii}| > \alpha R_i + (1-\alpha) S_i\} \;/\; \Phi$, then A is a nonsingular H-matrix.

Theorem 1. Let $A \quad (a_{ij})_{n\times n} \in C^{n\times n}$, for $\alpha \in \; 0,1$, if

$$|a_{ii}| > \frac{\alpha}{x_i} \sum_{j\neq i} |a_{ij}|\, x_j + \frac{1-\alpha}{y_i} \sum_{j\neq i} |a_{ji}|\, y_j, \; i \in \; ^{\alpha}_{1} \tag{1}$$

$$|a_{ii}| \geq \tfrac{\alpha}{x_i} \left(\sum_{j\in N_1^\alpha} |a_{ij}|\, x_j + \sum_{j\in N_2^\alpha, j\neq i} |a_{ij}| \right)$$

$$+\frac{1-\alpha}{y_i} \left(\sum_{j\in N_1^\alpha} |a_{ji}|\, y_j + \sum_{j\in N_2^\alpha, j\neq i} |a_{ji}| \right), \; i \in \; ^{\alpha}_{2} \tag{2}$$

where $0 < x_i < 1$, $0 < y_i < 1$, $i \in \langle n\rangle$. Then A is a nonsingular H-matrix.

Proof: Let

$$b_i \quad \frac{x_i y_i\,|a_{ii}| - y_i \alpha \sum_{j\neq i} |a_{ij}|\, x_j - (1-\alpha) \sum_{j\neq i} |a_{ji}|\, y_j x_i}{y_i \alpha \sum_{j\neq i} |a_{ij}|\, x_j + (1-\alpha) \sum_{j\neq i} |a_{ji}|\, y_j x_i}, \tag{3}$$

$i \in \; ^{\alpha}_{1}$.

From (1) we know that $0 < b_i < +\infty$. Let

$$c_i \quad b_i \frac{\sum_{j\neq i} |a_{ij}|\, x_j}{\sum_{j\in N_2^\alpha} |a_{ij}|} \quad, f_i \quad b_i \frac{\sum_{j\neq i} |a_{ji}|\, y_j}{\sum_{j\in N_2^\alpha} |a_{ji}|} \tag{4}$$

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when $\sum_{j\in N_2^\alpha}|a_{ij}| \quad 0, \sum_{j\in N_2^\alpha}|a_{ji}| \quad 0$, we denote $c_i \quad \infty,\ f_i \quad \infty$, according to the hypothesis of this paper, we have $c_i > 0,\ f_i > 0$.

We denote

$$^{\alpha}_{x} \quad \{i \in \ ^{\alpha}_{2}\,|x_i \quad 1\},\ \ ^{\alpha}_{y} \quad \{i \in \ ^{\alpha}_{2}\,|y_i \quad 1\}.$$

There must exits a small enough positive number ε, such that

$$0 < \varepsilon < \ \Big\{_{i\in N_1^\alpha}\{c_i\},\ _{i\in N_1^\alpha}\{f_i\},$$
$$_{i\in N_2^\alpha\setminus N_x^\alpha}\{1-x_i\},\ _{i\in N_2^\alpha\setminus N_y^\alpha}\{1-y_i\}\Big\}.$$

We choose positive diagonal matrix

$$diag\,(d_1, d_2, \cdots, d_n)$$

and

$$diag\,(e_1, e_2, \cdots, e_n)\,,$$

where

$$d_i \quad \begin{cases} x_i & i\in \ ^{\alpha}_{1} \\ x_i & i\in \ ^{\alpha}_{x} \\ x_i+\varepsilon & i\in \ ^{\alpha}_{2}\setminus \ ^{\alpha}_{x}\end{cases} \qquad e_i \quad \begin{cases} y_i & i\in \ ^{\alpha}_{1} \\ y_i & i\in \ ^{\alpha}_{y} \\ y_i+\varepsilon & i\in \ ^{\alpha}_{2}\setminus \ ^{\alpha}_{y}\end{cases}$$

In the follows, we just need to prove that $\quad (b_{ij})$ is a strictly α diagonally dominant matrix.

For $\forall i \in \ ^{\alpha}_{1}$, according to (3) we have

$$|a_{ij}|\,x_i y_i \quad (1+b_i)\left(\alpha\sum_{j\neq i}|a_{ij}|\,x_j y_i \right.$$
$$\left. +(1-\alpha)\sum_{j\neq i}|a_{ji}|\,y_j x_i\right). \qquad (5)$$

There will be four cases:

Case one: $\sum_{j\in N_2^\alpha}|a_{ij}| \quad \sum_{j\in N_2^\alpha}|a_{ji}| \quad 0$, according to(1)we have:

$$b_{ii} \quad y_i\,|a_{ii}|\,x_i > y_i\alpha\sum_{j\neq i}|a_{ij}|\,x_j + (1-\alpha)\sum_{j\neq i}|a_{ji}|\,y_j\cdot x_i$$
$$\alpha\sum_{j\in N_1^\alpha}|b_{ij}| + (1-\alpha)\sum_{j\in N_1^\alpha}|b_{ji}|$$
$$\alpha R_i\,(\quad) + (1-\alpha)\,S_i\,(\quad)\,.$$

Case two: $\sum_{j\in N_2^\alpha}|a_{ij}| \quad 0, \sum_{j\in N_2^\alpha}|a_{ji}| \ / \ 0$, in this case, $|a_{ij}| \quad 0$ for any $j \in \ ^{\alpha}_{2}$. according to (4) we have:

$$\varepsilon < f_i \Leftrightarrow \varepsilon\sum_{j\in N_2^\alpha}|a_{ji}| < b_i\sum_{j\neq i}|a_{ji}|y_j$$
$$\Leftrightarrow (1+b_i)\sum_{j\neq i}|a_{ji}|y_j > \sum_{j\neq i}|a_{ji}|y_j + \varepsilon\sum_{j\in N_2^\alpha}|a_{ji}|. \qquad (6)$$

With (5), (6) and the hypothesis of the paper, we have:

$$b_{ii} \quad y_i\,|a_{ii}|\,x_i \quad y_i\alpha\,(1+b_i)\sum_{j\neq i}|a_{ij}|\,x_j$$
$$+ (1-\alpha)\,(1+b_i)\sum_{j\neq i}|a_{ji}|\,y_j x_i$$
$$> y_i\alpha\sum_{\substack{j\in \ ^{\alpha}_{1} \\ j \ / \ i}}|a_{ij}|\,x_j$$
$$+ (1-\alpha)\left(\sum_{j\neq i}|a_{ji}|\,y_j + \varepsilon\sum_{j\in N_2^\alpha}|a_{ji}|\right)x_i$$
$$> y_i\alpha\sum_{\substack{j\in \ ^{\alpha}_{1} \\ j \ / \ i}}|a_{ij}|\,x_j + x_i\,(1-\alpha)\,\times$$
$$\left(\sum_{\substack{j\in \ ^{\alpha}_{1} \\ j \ / \ i}}|a_{ji}|\,y_j + \sum_{j\in N_y^\alpha}|a_{ji}|\,y_j + \sum_{j\in N_2^\alpha\setminus N_y^\alpha}|a_{ji}|\,(y_j+\varepsilon)\right)$$
$$\alpha R_i\,(\quad) + (1-\alpha)\,S_i\,(\quad)\,.$$

Case three: $\sum_{j\in N_2^\alpha}|a_{ij}| \ / \ 0, \sum_{j\in N_2^\alpha}|a_{ji}| \quad 0$. As the same proof of case two, we can obtain

$$|b_{ii}| > R_i\,(\quad)^{\alpha}\,C_i\,(\quad)^{1-\alpha}\,.$$

Case four: $\sum_{j\in N_2^\alpha}|a_{ij}| \ / \ 0, \sum_{j\in N_2^\alpha}|a_{ji}| \ / \ 0$ according to (4) we have:

$$\varepsilon < c_i \Leftrightarrow \varepsilon\sum_{j\in N_2^\alpha}|a_{ij}| < b_i\sum_{j\neq i}|a_{ij}|x_j$$
$$\Leftrightarrow (1+b_i)\sum_{j\neq i}|a_{ij}|x_j > \sum_{j\neq i}|a_{ij}|x_j + \varepsilon\sum_{j\in N_2^\alpha}|a_{ij}|.$$

From the above inequation and the inequation (6), we have

$$b_{ii} \quad y_i\,|a_{ii}|\,x_i \quad \alpha\,(1+b_i)\,y_i\sum_{j\neq i}|a_{ij}|\,x_j$$
$$+ (1-\alpha)\,(1+b_i)\,x_i\sum_{j\neq i}|a_{ji}|\,y_j$$
$$> \alpha y_i\left(\sum_{j\neq i}|a_{ij}|\,x_j + \varepsilon\sum_{j\in N_2^\alpha}|a_{ij}|\right)$$
$$+ (1-\alpha)\,x_i\left(\sum_{j\neq i}|a_{ji}|\,y_j + \varepsilon\sum_{j\in N_2^\alpha}|a_{ji}|\right)$$
$$\geq \alpha y_i\left(\sum_{j\in N_1^\alpha\bigcup N_y^\alpha}|a_{ij}|\,x_j + \sum_{j\in N_2^\alpha\setminus N_x^\alpha}(x_j+\varepsilon)\,|a_{ij}|\right)$$

$$+x_i\,(1-\alpha)\left(\sum_{j\in \mathrm{N}_1^\alpha \bigcup \mathrm{N}_y^\alpha} |a_{ji}|\,y_j + \sum_{j\in \mathrm{N}_2^\alpha\setminus \mathrm{N}_y^\alpha} (y_j+\varepsilon)\,|a_{ji}|\right)$$

$$\alpha R_i\,(\quad) + (1-\alpha)\,S_i\,(\quad)\,.$$

For any $i \in \ {}_2^\alpha$, from the choice of ε and the positive diagonal matrices D and E, we know that $0 < d_i, e_i \leq 1$, for any $i \in \ {}_2^\alpha$.

Case one: $i \in \ {}_x^\alpha \cap \ {}_y^\alpha$

$$|b_{ii}| \quad |a_{ii}| \geq \alpha\left(\sum_{j\in \mathrm{N}_1^\alpha} |a_{ij}|\,x_j + \sum_{\substack{j\in \ {}_2^\alpha \\ j\ /\ i}} |a_{ij}|\right)$$

$$+\,(1-\alpha)\left(\sum_{j\in \mathrm{N}_1^\alpha} |a_{ji}|\,y_j + \sum_{\substack{j\in \ {}_2^\alpha \\ j\ /\ i}} |a_{ji}|\right)$$

$$>\alpha\left(\sum_{j\in \langle n\rangle\setminus \mathrm{N}_x^\alpha} |a_{ij}|\,x_j + \sum_{\substack{j\in \ {}_x^\alpha \\ j\ /\ i}} |a_{ij}|\right)$$

$$+\,(1-\alpha)\left(\sum_{j\in \langle n\rangle\setminus \mathrm{N}_y^\alpha} |a_{ji}|\,y_j + \sum_{\substack{j\in \ {}_y^\alpha \\ j\ /\ i}} |a_{ji}|\right)$$

$$\alpha R_i\,(\quad) + (1-\alpha)\,S_i\,(\quad)\,.$$

Case two: $i \in \ {}_x^\alpha, i \notin \ {}_y^\alpha$, if $\alpha\ /\ 1$, from (2) we have

$$(y_i+\varepsilon)\,|a_{ii}|$$

$$\geq (y_i+\varepsilon)\,\alpha\left(\sum_{j\in \mathrm{N}_1^\alpha} |a_{ij}|\,x_j + \sum_{\substack{j\in \ {}_2^\alpha \\ j\ /\ i}} |a_{ij}|\right), \qquad (7)$$

$$+\,(1-\alpha)\left(\sum_{j\in \mathrm{N}_1^\alpha} |a_{ji}|\,y_j + \sum_{\substack{j\in \ {}_2^\alpha \\ j\ /\ i}} |a_{ji}|\right)$$

Hence

$$|b_{ii}| \quad (y_i+\varepsilon)\,|a_{ii}|$$

$$\geq (y_i+\varepsilon)\,\alpha\left(\sum_{j\in \mathrm{N}_1^\alpha} |a_{ij}|\,x_j + \sum_{\substack{j\in \ {}_2^\alpha \\ j\ /\ i}} |a_{ij}|\right)$$

$$+\,(1-\alpha)\left(\sum_{j\in \mathrm{N}_1^\alpha} |a_{ji}|\,y_j + \sum_{\substack{j\in \ {}_2^\alpha \\ j\ /\ i}} |a_{ji}|\right)$$

$$>(y_i+\varepsilon)\,\alpha\left(\sum_{j\in \mathrm{N}_1^\alpha} |a_{ij}|\,x_j + \sum_{j\in \mathrm{N}_2^\alpha\setminus \mathrm{N}_x^\alpha} |a_{ij}|\,(x_j+\varepsilon) + \sum_{j\in \mathrm{N}_x^\alpha} |a_{ij}|\right)$$

$$+\,(1-\alpha)\left(\sum_{j\in \mathrm{N}_1^\alpha} |a_{ji}|\,y_j + \sum_{\substack{j\in \mathrm{N}_2^\alpha\setminus \mathrm{N}_y^\alpha \\ j\neq i}} |a_{ji}|\,(y_j+\varepsilon) + \sum_{j\in \mathrm{N}_y^x} |a_{ji}|\right)$$

$$\alpha R_i\,(\quad) + (1-\alpha)\,S_i\,(\quad)\,.$$

Case three: $i \notin \ {}_x^\alpha, i \in \ {}_y^\alpha$, as the same proof of case two, we can obtain

$$|b_{ii}| > \alpha R_i\,(\quad) + (1-\alpha)\,S_i\,(\quad)\,.$$

Case four: $i \notin \ {}_x^\alpha, i \notin \ {}_y^\alpha$, from (2) we have

$$(y_i+\varepsilon)\,|a_{ii}|\,(x_i+\varepsilon)$$

$$\geq (y_i+\varepsilon)\,\alpha\left(\sum_{j\in \mathrm{N}_1^\alpha} |a_{ij}|\,x_j + \sum_{\substack{j\in \ {}_2^\alpha \\ j\ /\ i}} |a_{ij}|\right)$$

$$+\,(1-\alpha)\left(\sum_{j\in \mathrm{N}_1^\alpha} |a_{ji}|\,y_j + \sum_{\substack{j\in \ {}_2^\alpha \\ j\ /\ i}} |a_{ji}|\right)(x_i+\varepsilon)$$

Since

$$(y_i+\varepsilon)\,\alpha\left(\sum_{j\in \mathrm{N}_1^\alpha} |a_{ij}|\,x_j + \sum_{\substack{j\in \ {}_2^\alpha \\ j\ /\ i}} |a_{ij}|\right) > \alpha R_i\,(\quad)\,,$$

$$(1-\alpha)\left(\sum_{j\in \mathrm{N}_1^\alpha} |a_{ji}|\,y_j + \sum_{\substack{j\in \ {}_2^\alpha \\ j\ /\ i}} |a_{ji}|\right)(x_i+\varepsilon)$$

$$> (1-\alpha)\, S_i(\ \ ),$$

we have

$$|b_{ii}| \quad (y_i+\varepsilon)\,|a_{ii}|\,(x_i+\varepsilon)$$

$$\geq (y_i+\varepsilon)\,\alpha \left(\sum_{j\in \mathrm{N}_1^{\alpha}} |a_{ij}|\, x_j + \sum_{\substack{j\in \ \ _2^{\alpha} \\ j\ /\ i}} |a_{ij}| \right)$$

$$+ (1-\alpha) \left(\sum_{j\in \mathrm{N}_1^{\alpha}} |a_{ji}|\, y_j + \sum_{\substack{j\in \ \ _2^{\alpha} \\ j\ /\ i}} |a_{ji}| \right) (x_i+\varepsilon)$$

$$> \alpha R_i(\ \ ) + (1-\alpha)\, S_i(\ \ ).$$

We see that for any $i \in \langle n \rangle$, we have $|b_{ii}| > \alpha R_i(\ \ ) + (1-\alpha)\, S_i(\ \ )$. According to Lemma 1, we know that matrix B is a nonsingular H-matrix, so matrix A is a nonsingular H-matrix.

Let $A \quad (a_{ij})_{n\times n} \in C^{n\times n}$, $0 < x_i,\ y_i < 1,\ i \in \langle n \rangle$ satisfy the equation (2), we denote

$$K_\alpha \quad \left\{ i \in <n> \,\middle|\, |a_{ii}| > \frac{\alpha}{x_i} \sum_{j\neq i} |a_{ij}|\, x_j + \frac{1-\alpha}{y_i} \sum_{j\neq i} |a_{ji}|\, y_j \right\}.$$

Theorem 2 Let $A \quad (a_{ij})_{n\times n} \in C^{n\times n}$, for $\alpha \in \ 0,1\ $, if $0 < x_i < 1,\ 0 < y_i < 1,\ i \in \langle n \rangle$ satisfy the inequations (2) and

$$|a_{ii}| \geq \frac{\alpha}{x_i} \sum_{j\neq i} |a_{ij}|\, x_j + \frac{1-\alpha}{y_i} \sum_{j\neq i} |a_{ji}|\, y_j,\ i \in \ \ _1^{\alpha} \tag{8}$$

and $K_\alpha\ /\ \emptyset$, for any $i_0 \in (<n> \backslash K_\alpha)$, there exists a nonzero elements chain $a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{k-1} i_k}\ /\ 0$ such that $i_k \in K_\alpha$, then A is a nonsingular H-matrix.

Proof: we structure two positive diagonal matrices: D $diag\,(x_1, x_2, \cdots, x_n)$ and $\quad diag\,(y_1, y_2, \cdots, y_n)$, and notes $B \quad (b_{ij}) \quad EAD$. So for any $i \in <n>$, we have

$$|b_{ii}| \geq \alpha R_i(B) + (1-\alpha)\, S_i(B).$$

Obviously, K_α can be note

$$K_\alpha \quad \{ i \in <n> \,|\, |b_{ii}| > \alpha R_i(B) + (1-\alpha)\, S_i(B) \},$$

for any $i_0 \in K_\alpha$, we have $b_{i_0 i_1} b_{i_1 i_2} \cdots b_{i_{k-1} i_k}\ /\ 0$ such that $i_k \in K_\alpha$. So according to Lemma 2, we know that matrix B is a nonsingular H-matrix, so matrix A is a nonsingular H-matrix.

From Theorem 2, we can get the following corollary.

Corollary Let $A \quad (a_{ij})_{n\times n} \in C^{n\times n}$ be irreducible, for $\alpha \in (0,1)$, if $0 < x_i < 1,\ 0 < y_i < 1,\ i \in \langle n \rangle$ satisfy the inequations (2) and (8), $\widetilde{I}\ /\ \emptyset$, where

$$\widetilde{I} \quad \left\{ v \in S(A) \,\middle|\, \ _{i\in v} y_i\, |a_{ii}|\, x_i\ /\ \ _{i\in v} \widetilde{R}_i(A) \right.$$
$$\left. or\ \ _{i\in v} y_i\, |a_{ii}|\, x_i\ /\ \ _{i\in v} \widetilde{C}_i(A) \right\},$$

$$\widetilde{R}_i(A) \quad y_i \sum_{j\neq i} |a_{ij}| x_j, \quad \widetilde{C}_i(A) \quad x_i \sum_{j\neq i} |a_{ji}| y_j,$$

then A is a nonsingular H-matrix.

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