

# On Cross-Ratio in some Moufang-Klingenberg Planes

Atila Akpinar and Basri Celik

**Abstract**—In this paper we are interested in Moufang-Klingenberg planes  $M(\mathcal{A})$  defined over a local alternative ring  $\mathcal{A}$  of dual numbers. We show that a collineation of  $M(\mathcal{A})$  preserve cross-ratio. Also, we obtain some results about harmonic points.

**Keywords**—Moufang-Klingenberg planes, local alternative ring, projective collineation, cross-ratio, harmonic points.

## I. INTRODUCTION

In the Euclidean plane, Desargues established the fundamental fact that cross-ratio (a concept originally introduced by Pappus of Alexandria c.300 B.C) is invariant under projection [4, p. 133]. For this reason, cross-ratio is one of the most important concepts of projective geometry.

In this paper we deal with the class (which we will denote by  $M(\mathcal{A})$ ) of Moufang-Klingenberg (MK) planes coordinatized by a local alternative ring

$$\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$$

(an alternative field  $\mathbf{A}$ ,  $\varepsilon \notin \mathbf{A}$  and  $\varepsilon^2 = 0$ ) introduced by Blunck in [8]. We will show that a collineation of  $M(\mathcal{A})$  given in [2] preserves cross-ratio. Moreover, we will obtain some results related to harmonic points. For more information about some well-known properties of cross-ratio in the case of Moufang planes or MK-planes  $M(\mathcal{A})$ , respectively, it can be seen the papers of [10], [5], [9] or [8], [1].

The paper is organized as follows: Section 2 includes some basic definitions and results from the literature. In Section 3 we will give a collineation of  $M(\mathcal{A})$  from [2] and we show that this collineation preserves cross-ratio. Finally, we obtain some results on harmonic points.

## II. PRELIMINARIES

Let  $M = (\mathbf{P}, \mathbf{L}, \in, \sim)$  consist of an incidence structure  $(\mathbf{P}, \mathbf{L}, \in)$  (points, lines, incidence) and an equivalence relation ‘ $\sim$ ’ (neighbour relation) on  $\mathbf{P}$  and on  $\mathbf{L}$ , respectively. Then  $M$  is called a *projective Klingenberg plane* (PK-plane), if it satisfies the following axioms:

(PK1) If  $P, Q$  are non-neighbour points, then there is a unique line  $PQ$  through  $P$  and  $Q$ .

(PK2) If  $g, h$  are non-neighbour lines, then there is a unique point  $g \cap h$  on both  $g$  and  $h$ .

(PK3) There is a projective plane  $M^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$  and an incidence structure epimorphism  $\Psi : M \rightarrow M^*$ , such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q, \Psi(g) = \Psi(h) \Leftrightarrow g \sim h$$

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hold for all  $P, Q \in \mathbf{P}$ ,  $g, h \in \mathbf{L}$ .

A point  $P \in \mathbf{P}$  is called *near* a line  $g \in \mathbf{L}$  iff there exists a line  $h \sim g$  such that  $P \in h$ .

Let  $h, k \in \mathbf{L}$ ,  $C \in \mathbf{P}$ ,  $C$  is not symmetric to  $h$  and  $k$ . Then the well-defined bijection

$$\sigma := \sigma_C(k, h) : \begin{cases} h \rightarrow k \\ X \rightarrow XC \cap k \end{cases}$$

mapping  $h$  to  $k$  is called a *perspectivity* from  $h$  to  $k$  with center  $C$ . A product of a finite number of perspectivities is called a *projectivity*.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a *collineation* of  $M$ .

A *Moufang-Klingenberg plane* (MK-plane) is a PK-plane  $M$  that generalizes a Moufang plane, and for which  $M^*$  is a Moufang plane (for the exact definition see [3]).

An *alternative ring (field)*  $\mathbf{R}$  is a not necessarily associative ring (field) that satisfies the alternative laws

$$a(ab) = a^2b, (ba)a = ba^2, \forall a, b \in \mathbf{R}.$$

An alternative ring  $\mathbf{R}$  with identity element 1 is called *local* if the set  $\mathbf{I}$  of its non-unit elements is an ideal.

We are now ready to give consecutively two important lemmas related to alternative rings.

*Lemma 2.1:* The subring generated by any two elements of an alternative ring is associative (cf. [12, Theorem 3.1]).

*Lemma 2.2:* The identities

$$\begin{aligned} x(y(xz)) &= (xyx)z \\ ((yx)z)x &= y(xzx) \\ (xy)(zx) &= x(yz)x \end{aligned}$$

which are known as Moufang identities are satisfied in every alternative ring (cf. [11, p. 160]).

We summarize some basic concepts about the coordinatization of MK-planes from [3].

Let  $\mathbf{R}$  be a local alternative ring. Then  $M(\mathbf{R}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$  is the incidence structure with neighbour relation defined

as follows:

$$\begin{aligned}
 \mathbf{P} &= \{(x, y, 1) : x, y \in \mathbf{R}\} \\
 &\cup \{(1, y, z) : y \in \mathbf{R}, z \in \mathbf{I}\} \\
 &\cup \{(w, 1, z) : w, z \in \mathbf{I}\}, \\
 \mathbf{L} &= \{[m, 1, p] : m, p \in \mathbf{R}\} \\
 &\cup \{[1, n, p] : p \in \mathbf{R}, n \in \mathbf{I}\} \\
 &\cup \{[q, n, 1] : q, n \in \mathbf{I}\} \\
 [m, 1, p] &= \{(x, xm + p, 1) : x \in \mathbf{R}\} \\
 &\cup \{(1, zp + m, z) : z \in \mathbf{I}\} \\
 [1, n, p] &= \{(yn + p, y, 1) : y \in \mathbf{R}\} \\
 &\cup \{(zp + n, 1, z) : z \in \mathbf{I}\} \\
 [q, n, 1] &= \{(1, y, yn + q) : y \in \mathbf{R}\} \\
 &\cup \{(w, 1, wq + n) : w \in \mathbf{I}\}
 \end{aligned}$$

and

$$\begin{aligned}
 P = (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = Q &\Leftrightarrow \\
 x_i - y_i \in \mathbf{I} \ (i = 1, 2, 3), \forall P, Q \in \mathbf{P} & \\
 g = [x_1, x_2, x_3] \sim [y_1, y_2, y_3] = h &\Leftrightarrow \\
 x_i - y_i \in \mathbf{I} \ (i = 1, 2, 3), \forall g, h \in \mathbf{L}. &
 \end{aligned}$$

Now it is time to give the following theorem from [3].

*Theorem 2.1:*  $\mathbf{M}(\mathbf{R})$  is an MK-plane, and each MK-plane is isomorphic to some  $\mathbf{M}(\mathbf{R})$ .

Let  $\mathbf{A}$  be an alternative field and  $\varepsilon \notin \mathbf{A}$ . Consider  $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$  with componentwise addition and multiplication as follows:

$$(a_1 + a_2\varepsilon)(b_1 + b_2\varepsilon) = a_1b_1 + (a_1b_2 + a_2b_1)\varepsilon,$$

where  $a_i, b_i \in \mathbf{A}$  for  $i = 1, 2$ . Then  $\mathcal{A}$  is a local alternative ring with ideal  $\mathbf{I} = \mathbf{A}\varepsilon$  of non-units. The set of formal inverses of the non-units of  $\mathcal{A}$  is denoted as  $\mathbf{I}^{-1}$ . Calculations with the elements of  $\mathbf{I}^{-1}$  are defined as follows [7]:

$$\begin{aligned}
 (a\varepsilon)^{-1} + t &:= (a\varepsilon)^{-1} := t + (a\varepsilon)^{-1} \\
 q(a\varepsilon)^{-1} &:= (aq^{-1}\varepsilon)^{-1} \\
 (a\varepsilon)^{-1}q &:= (q^{-1}a\varepsilon)^{-1} \\
 \left((a\varepsilon)^{-1}\right)^{-1} &:= a\varepsilon,
 \end{aligned}$$

where  $(a\varepsilon)^{-1} \in \mathbf{I}^{-1}$ ,  $t \in \mathcal{A}$ ,  $q \in \mathcal{A} \setminus \mathbf{I}$ . (Other terms are not defined.). For more information about  $\mathcal{A}$  and its relation to MK-planes, the reader is referred to the papers of Blunck [7], [8]. In [8], the centre  $\mathbf{Z}(\mathcal{A})$  is defined to be the (commutative, associative) subring of  $\mathcal{A}$  which is commuting and associating with all elements of  $\mathcal{A}$ . It is  $\mathbf{Z}(\mathcal{A}) := \mathbf{Z}(\varepsilon) = \mathbf{Z} + \mathbf{Z}\varepsilon$ , where  $\mathbf{Z} = \{z \in \mathbf{A} : za = az, \forall a \in \mathbf{A}\}$  is the centre of  $\mathbf{A}$ . If  $\mathbf{A}$  is not associative, then  $\mathbf{A}$  is a Cayley division algebra over its centre  $\mathbf{Z}$ .

Throughout this paper we assume  $\text{char}\mathbf{A} \neq 2$  and we restrict ourselves to the MK-planes  $\mathbf{M}(\mathcal{A})$

Blunck [8] gives the following algebraic definition of the cross-ratio for the points on the line  $g := [1, 0, 0]$  in  $\mathbf{M}(\mathcal{A})$ .

$$\begin{aligned}
 (A, B; C, D) &:= (a, b; c, d) \\
 &= \langle \left( (a-d)^{-1}(b-d) \right) \left( (b-c)^{-1}(a-c) \right) \rangle > \\
 (Z, B; C, D) &:= (z^{-1}, b; c, d) \\
 &= \langle \left( (1-dz)^{-1}(b-d) \right) \left( (b-c)^{-1}(1-cz) \right) \rangle > \\
 (A, Z; C, D) &:= (a, z^{-1}; c, d) \\
 &= \langle \left( (a-d)^{-1}(1-dz) \right) \left( (1-cz)^{-1}(a-c) \right) \rangle > \\
 (A, B; Z, D) &:= (a, b; z^{-1}, d) \\
 &= \langle \left( (a-d)^{-1}(b-d) \right) \left( (1-zb)^{-1}(1-za) \right) \rangle > \\
 (A, B; C, Z) &:= (a, b; c, z^{-1}) \\
 &= \langle \left( (1-za)^{-1}(1-zb) \right) \left( (b-c)^{-1}(a-c) \right) \rangle >,
 \end{aligned}$$

where  $A = (0, a, 1), B = (0, b, 1), C = (0, c, 1), D = (0, d, 1), Z = (0, 1, z)$  are pairwise non-neighbour points of  $g$  and  $\langle x \rangle = \{y^{-1}xy : y \in \mathcal{A}\}$ .

In [7, Theorem 2], it is shown that the transformations

$$\begin{aligned}
 t_u(x) &= x + u; u \in \mathcal{A} \\
 r_u(x) &= xu; u \in \mathcal{A} \setminus \mathbf{I} \\
 i(x) &= x^{-1} \\
 l_u(x) &= ux = (ir_u^{-1}i)(x); u \in \mathcal{A} \setminus \mathbf{I}
 \end{aligned}$$

which are defined on the line  $g$  preserve cross-ratios. In [6, Corollary (iii)], it is also shown that the group generated by these transformations, which is denoted by  $\Lambda$ , equals to the group of projectivities of a line in  $\mathbf{M}(\mathcal{A})$ . The elements preserving cross-ratio of the group  $\Lambda$  defined on  $g$  will act a very important role in the proof of Theorem 3.1.

We give the following result from [1, Theorem 8]. This result states a simple way for calculation of the cross-ratio of the points on any line in  $\mathbf{M}(\mathcal{A})$ .

*Theorem 2.2:* Let  $\{O, U, V, E\}$  be the basis of  $\mathbf{M}(\mathcal{A})$  where  $O = (0, 0, 1), U = (1, 0, 0), V = (0, 1, 0), E = (1, 1, 1)$  (see [3, Section 4]). Then, according to types of lines, the cross-ratio of the points on the line  $l$  can be calculated as follows:

If  $A, B, C, D, Z$  are the pairwise non-neighbour points

- (a) of the line  $l = [m, 1, k]$ , where  $A = (a, am + k, 1), B = (b, bm + k, 1), C = (c, cm + k, 1), D = (d, dm + k, 1)$  are not near to the line  $UV = [0, 0, 1]$  and  $Z = (1, m + zp, z)$  is near to  $UV$ ;
- (b) of the line  $l = [1, n, p]$ , where  $A = (an + p, a, 1), B = (bn + p, b, 1), C = (cn + p, c, 1), D = (dn + p, d, 1)$  are not neighbour to  $V$  and  $Z = (n + zp, 1, z) \sim V$ ;
- (c) of the line  $l = [q, n, 1]$ , where  $A = (1, a, q + an), B = (1, b, q + bn), C = (1, c, q + cn), D = (1, d, q + dn)$  are not neighbour to  $V$  and  $Z = (z, 1, zq + n) \sim V$ ;

then

$$\begin{aligned} (A, B; C, D) &= (a, b; c, d) \\ (Z, B; C, D) &= (z^{-1}, b; c, d) \\ (A, Z; C, D) &= (a, z^{-1}; c, d) \\ (A, B; Z, D) &= (a, b; z^{-1}, d) \\ (A, B; C, Z) &= (a, b; c, z^{-1}). \end{aligned}$$

We can give an important theorem, from [1, Theorem 9], about cross-ratio.

**Theorem 2.3:** In  $M(A)$ , perspectivities preserve cross-ratios.

Now we give a definition in  $M(A)$ , well known from the case of Moufang planes [10]. In  $M(A)$ , any pairwise non-neighbour four points  $A, B, C, D \in l$  are called as *harmonic* if  $(A, B; C, D) = \langle -1 \rangle$  and we let  $h(A, B, C, D)$  represent the statement:  $A, B, C, D$  are harmonic.

### III. ON CROSS-RATIO IN $M(A)$ .

In this section we will give a collineation of  $M(A)$ , from [2]. Next, we show that the collineation preserve cross-ratios. Now we start with giving the collineation of  $M(A)$ , where  $w, z, q, n \in A$ : For any  $s \notin I$ , the map  $J_s$  transforms points and lines as follows:

$$\begin{aligned} (x, y, 1) &\rightarrow (ys^{-1}, xs, 1) \\ (1, y, z\varepsilon) &\rightarrow (1, sy^{-1}s, s(y^{-1}z)) \text{ if } y \notin I \\ (1, y, z\varepsilon) &\rightarrow (s^{-1}ys^{-1}, 1, s^{-1}z) \text{ if } y \in I \\ (w\varepsilon, 1, z\varepsilon) &\rightarrow (1, sws, sz) \end{aligned}$$

and

$$\begin{aligned} [m, 1, k] &\rightarrow [sm^{-1}s, 1, -(km^{-1})s] \text{ if } m \notin I \\ [m, 1, k] &\rightarrow [1, s^{-1}ms^{-1}, ks^{-1}] \text{ if } m \in I \\ [1, n\varepsilon, p] &\rightarrow [sns, 1, ps] \\ [q\varepsilon, n\varepsilon, 1] &\rightarrow [sn, s^{-1}q, 1]. \end{aligned}$$

Now we are ready to give the following

**Theorem 3.1:** The collineation  $J_s$  preserve cross-ratio.

*Proof:* Let  $A, B, C, D$  and  $Z$  be the points with the property given in the statement of Theorem 2.2. Then, it is obvious that

$$\begin{aligned} (A, B; C, D) &= (a, b; c, d) \\ (Z, B; C, D) &= (z^{-1}, b; c, d) \\ (A, Z; C, D) &= (a, z^{-1}; c, d) \\ (A, B; Z, D) &= (a, b; z^{-1}, d) \\ (A, B; C, Z) &= (a, b; c, z^{-1}), \end{aligned} \tag{1}$$

where  $z \in I$ . In this case we must find the effect of  $\varphi$  to the points of any line where  $\varphi$  is the collineations  $J_s$ .

Let  $\varphi = J_s$ . If  $l = [m, 1, k]$ , then

$$\begin{aligned} \varphi(X) &= \varphi(x, xm + k, 1) \\ &= ((xm + k)s^{-1}, xs, 1) \\ \varphi(Z) &= \varphi(1, m + zk, z) \\ &= (1, s(m + zk)^{-1}s, s((m + zk)^{-1}z)) \\ &\text{for } m + zk \notin I \\ \varphi(Z) &= \varphi(1, m + zk, z) \\ &= (s^{-1}(m + zk)s^{-1}, 1, s^{-1}z), \\ &\text{for } m + zk \in I \\ \varphi(l) &= [sm^{-1}s, 1, -(km^{-1})s] \text{ for } m \notin I \\ \varphi(l) &= [1, s^{-1}ms^{-1}, ks^{-1}] \text{ for } m \in I. \end{aligned}$$

In this case, from (a) of Theorem 2.2, the cross-ratio of the points of  $[sm^{-1}s, 1, -(km^{-1})s]$  is as follows:

$$\begin{aligned} (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= ((am + k)s^{-1}, (bm + k)s^{-1}; \\ &(cm + k)s^{-1}, (dm + k)s^{-1}) \\ &= {}^\sigma(a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left( \left( s \left( (m + zk)^{-1}z \right) \right)^{-1}, (bm + k)s^{-1}; \right. \\ &(cm + k)s^{-1}, (dm + k)s^{-1} \left. \right) \\ &= {}^\sigma(z^{-1}, b; c, d), \end{aligned}$$

where  $\sigma = r_{m^{-1}} \circ t_{-k} \circ r_s \in \Lambda$ . From (b) of Theorem 2.2, the cross-ratio of the points of  $[1, s^{-1}ms^{-1}, ks^{-1}]$  is as follows:

$$\begin{aligned} (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (as, bs; cs, ds) = {}^\sigma(a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (z^{-1}s, bs; cs, ds) = {}^\sigma(z^{-1}, b; c, d), \end{aligned}$$

where  $\sigma = r_{s^{-1}} \in \Lambda$ .

If  $l = [1, n, p]$ , then

$$\begin{aligned} \varphi(X) &= \varphi(xn + p, x, 1) = (xs^{-1}, (xn + p)s, 1) \\ \varphi(Z) &= \varphi(n + zp, 1, z) = (1, s(n + zp)s, sz) \end{aligned}$$

and

$$\varphi(l) = [sns, 1, ps].$$

In this case, from (a) of Theorem 2.2, the cross-ratio of the points of  $[sns, 1, ps]$  is as follows:

$$\begin{aligned} (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (as^{-1}, bs^{-1}; cs^{-1}, ds^{-1}) = {}^\sigma(a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (z^{-1}s^{-1}, bs^{-1}; cs^{-1}, ds^{-1}) = {}^\sigma(z^{-1}, b; c, d), \end{aligned}$$

where  $\sigma = r_s \in \Lambda$ . If  $l = [q, n, 1]$ , then

$$\begin{aligned} \varphi(X) &= \varphi(1, x, q + xn) = (1, sx^{-1}s, s(x^{-1}(q + xn))) \\ &\text{for } x \notin I \\ \varphi(X) &= \varphi(1, x, q + xn) = (s^{-1}xs^{-1}, 1, s^{-1}(q + xn)) \\ &\text{for } x \in I \\ \varphi(Z) &= \varphi(z, 1, zq + n) = (1, szs, s(zq + n)) \end{aligned}$$

and

$$\varphi(l) = [sn, s^{-1}q, 1].$$

In this case, from (c) of Theorem 2.2, the cross-ratio of the points of  $[sn, s^{-1}q, 1]$  is as follows:

$$\begin{aligned} &(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) \\ &= (sa^{-1}s, sb^{-1}s; sc^{-1}s, sd^{-1}s) \\ &=^\sigma (a, b; c, d), \\ &(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) \\ &= (szs, sb^{-1}s; sc^{-1}s, sd^{-1}s) \\ &=^\sigma (z^{-1}, b; c, d), \end{aligned}$$

where  $\sigma = i \circ l_{s^{-1}} \circ r_{s^{-1}} \in \Lambda$ . Consequently, by considering other all cases we get

$$\begin{aligned} (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (z^{-1}, b; c, d) \\ (\varphi(A), \varphi(Z); \varphi(C), \varphi(D)) &= (a, z^{-1}; c, d) \\ (\varphi(A), \varphi(B); \varphi(Z), \varphi(D)) &= (a, b; z^{-1}, d) \\ (\varphi(A), \varphi(B); \varphi(C), \varphi(Z)) &= (a, b; c, z^{-1}) \end{aligned}$$

for collineation  $\varphi$ . Combining the last result and the result of (1), the proof is completed. ■

Now we are ready to give the other results of the paper. On  $\mathcal{A}$  we give the following theorem, an alternate definition of harmonicity and given for an alternative ring  $\mathbf{A}$  with  $\text{char} \mathbf{A} \neq 2$ .

**Theorem 3.2:** Let  $a, b, c, d \in \mathcal{A}$ . Then  $h(a, b, c, d)$  if and only if

- 1) if  $a, b, c, d \in \mathcal{A}$ ,  $2(a-b)^{-1} = (a-c)^{-1} + (a-d)^{-1}$ .
- 2) if  $a = z^{-1}$ ,  $2(d-c)^{-1} + (c-b)^{-1} = z \in \mathbf{I}$ .
- 3) if  $b = z^{-1}$ ,  $2(c-d)^{-1} + (d-a)^{-1} = z \in \mathbf{I}$ .
- 4) if  $c = z^{-1}$ ,  $2(b-a)^{-1} + (d-b)^{-1} = z \in \mathbf{I}$ .
- 5) if  $d = z^{-1}$ ,  $2(a-b)^{-1} + (c-a)^{-1} = z \in \mathbf{I}$ .

*Proof:* 1. From the definition of cross-ratio,

$$h(a, b, c, d) = \left( (a-d)^{-1}(b-d) \right) \left( (b-c)^{-1}(a-c) \right) = -1.$$

By direct computation (with Lemma 2.1),

$$\begin{aligned} (a-d)^{-1}(b-d) &= -(a-c)^{-1}(b-c) \\ (a-d)^{-1}(b-a+a-d) &= -(a-c)^{-1}(b-a+a-c) \\ (a-d)^{-1}(b-a)+1 &= -(a-c)^{-1}(b-a)-1 \\ 2 &= -(a-c)^{-1}(b-a) - (a-d)^{-1}(b-a) \\ 2(a-b)^{-1} &= (a-c)^{-1} + (a-d)^{-1}. \end{aligned}$$

2. From the definition of cross-ratio,

$$\begin{aligned} &h(z^{-1}, b, c, d) \\ &= \left( (1-dz)^{-1}(b-d) \right) \left( (b-c)^{-1}(1-cz) \right) = -1. \end{aligned}$$

By direct computation (Lemma 2.1),

$$\begin{aligned} (b-c)^{-1}(1-cz) &= -(b-d)^{-1}(1-dz) \\ (b-c)^{-1}(1-cz) &= -(b-d)^{-1}(1-cz+cz-dz) \\ (b-c)^{-1}(1-cz) &= -(b-d)^{-1}(1-cz) \\ &\quad - (b-d)^{-1}((c-d)z) \\ \left( (b-c)^{-1} + (b-d)^{-1} \right) (1-cz) &= -(b-d)^{-1}((c-d)z) \\ (b-c)^{-1} + (b-d)^{-1} &= - \left( (b-d)^{-1}((c-d)z) \right) (1+cz) \\ (b-c)^{-1} + (b-d)^{-1} &= -(b-d)^{-1}((c-d)z) \\ (b-d)(b-c)^{-1} + 1 &= -(c-d)z \\ (b-c+c-d)(b-c)^{-1} + 1 &= -(c-d)z \\ 2 + (c-d)(b-c)^{-1} &= -(c-d)z \\ 2(c-d)^{-1} + (b-c)^{-1} &= -z \\ 2(d-c)^{-1} + (c-b)^{-1} &= z \in \mathbf{I}, \end{aligned}$$

where  $zz = 0$  since  $z \in \mathbf{I}$ .

3. The proof is same the proof of 2.

4. From the definition of cross-ratio,

$$\begin{aligned} &h(a, b, z^{-1}, d) \\ &= \left( (a-d)^{-1}(b-d) \right) \left( (1-zb)^{-1}(1-za) \right) = -1. \end{aligned}$$

By direct computation (Lemma 2.1),

$$\begin{aligned} (1-zb)^{-1}(1-za) &= -(b-d)^{-1}(a-d) \\ (1+zb)(1-za) &= -(b-d)^{-1}(a-b+b-d) \\ 1+zb-za &= -(b-d)^{-1}(a-b)-1 \\ 2+z(b-a) &= -(b-d)^{-1}(a-b) \\ 2(b-a)^{-1} + z &= (b-d)^{-1} \\ 2(b-a)^{-1} + (d-b)^{-1} &= z \in \mathbf{I}, \end{aligned}$$

where  $(1-zb)^{-1} = 1+zb$  and  $zz = 0$ .

5. The proof is same the proof of 4. ■

Now, we give the following theorem, given as without proof in [10] for  $\mathbf{A}$ .

**Theorem 3.3:** On  $\mathcal{A}$ , the followings is valid:

- 1)  $h\left(0, a, 0^{-1}, \frac{a}{2}\right)$
- 2)  $h\left(a, b, 0^{-1}, \frac{a+b}{2}\right)$
- 3)  $h\left(a, -a, 0^{-1}, 0\right)$
- 4)  $h\left(1, -1, a, a^{-1}\right)$
- 5)  $h\left(a^2, 1, a, -a\right)$

*Proof:* 1. By the definition of cross-ratio, since

$$\left(0, a, 0^{-1}, \frac{a}{2}\right) = \left(0 - \frac{a}{2}\right)^{-1} \left(a - \frac{a}{2}\right) = \frac{-2a}{a} \frac{a}{2} = -1,$$

then  $h\left(0, a, 0^{-1}, \frac{a}{2}\right)$ .

2. By the definition of cross-ratio, since

$$\begin{aligned} \left(a, b, 0^{-1}, \frac{a+b}{2}\right) &= \left(a - \frac{a+b}{2}\right)^{-1} \left(b - \frac{a+b}{2}\right) \\ &= \left(\frac{a-b}{2}\right)^{-1} \left(\frac{b-a}{2}\right) = -1, \end{aligned}$$

then  $h(a, b, 0^{-1}, \frac{a+b}{2})$ .

3. By the definition of cross-ratio, since

$$(a, -a, 0^{-1}, 0) = (a - 0)^{-1} (-a - 0) = -1,$$

then  $h(a, -a, 0^{-1}, 0)$ .

4. By the definition of cross-ratio, since

$$\begin{aligned} (1, -1, a, a^{-1}) &= \left( (1 - a^{-1})^{-1} (-1 - a^{-1}) \right) \\ &\quad \left( (-1 - a)^{-1} (1 - a) \right) \\ &= \left( (a^{-1} - 1)^{-1} - (1 - a^{-1})^{-1} a^{-1} \right) \\ &\quad \left( (-1 - a)^{-1} + (1 + a)^{-1} a \right) \\ &= \left( (a^{-1} - 1)^{-1} - (a(1 - a^{-1}))^{-1} \right) \\ &\quad \left( (-1 - a)^{-1} + (a^{-1}(1 + a))^{-1} \right) \\ &= \left( (a^{-1} - 1)^{-1} - (a - 1)^{-1} \right) \\ &\quad \left( - (1 + a)^{-1} + (a^{-1} + 1)^{-1} \right) \\ &= (a^{-1} - 1)^{-1} (a^{-1} + 1)^{-1} - (1 + a)^{-1} \\ &\quad - (a - 1)^{-1} \left( (a^{-1} + 1)^{-1} - (1 + a)^{-1} \right) \\ &= (a^{-1} - 1)^{-1} (a^{-1} + 1)^{-1} - (a^{-1} - 1)^{-1} \\ &\quad (1 + a)^{-1} - (a - 1)^{-1} (a^{-1} + 1)^{-1} \\ &\quad + (a - 1)^{-1} (1 + a)^{-1} \\ &= \left( (a^{-1} + 1) (a^{-1} - 1) \right)^{-1} \\ &\quad - \left( (1 + a) (a^{-1} - 1) \right)^{-1} \\ &\quad - \left( (a^{-1} + 1) (a - 1) \right)^{-1} \\ &\quad + \left( (1 + a) (a - 1) \right)^{-1} \\ &= (a^{-1} a^{-1} - a^{-1} + a^{-1} - 1)^{-1} \\ &\quad - (a^{-1} - 1 + 1 - a)^{-1} \\ &\quad - (1 - a^{-1} + a - 1)^{-1} + (a - 1 + aa - a)^{-1} \\ &= (a^{-1} a^{-1} - 1)^{-1} - (a^{-1} - a)^{-1} \\ &\quad - (-a^{-1} + a)^{-1} + (-1 + aa)^{-1} \\ &= (a^{-1} (a^{-1} - a))^{-1} - (a^{-1} - a)^{-1} \\ &\quad + (a^{-1} - a)^{-1} - (a (a^{-1} - a))^{-1} \\ &= (a^{-1} - a)^{-1} a - (a^{-1} - a)^{-1} a^{-1} \\ &= (a^{-1} - a)^{-1} (a - a^{-1}) \\ &= -1, \end{aligned}$$

then  $h(1, -1, a, a^{-1})$ .

5. By the definition of cross-ratio, since

$$\begin{aligned} (a^2, 1, a, -a) &= \left( (a^2 + a)^{-1} (1 + a) \right) \left( (1 - a)^{-1} (a^2 - a) \right) \\ &= \left( ((a + 1)a)^{-1} (1 + a) \right) \left( (1 - a)^{-1} ((a - 1)a) \right) \\ &= \left( a^{-1} (a + 1)^{-1} (1 + a) \right) \left( (1 - a)^{-1} (a - 1)a \right) \\ &= a^{-1} (-a) \\ &= -1, \end{aligned}$$

then  $h(a^2, 1, a, -a)$ . ■

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