

Induced Graphoidal Covers in a Graph

K. Ratan Singh, P. K. Das

Abstract—An induced graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G such that every path in ψ has at least two vertices, every vertex of G is an internal vertex of at most one path in ψ , every edge of G is in exactly one path in ψ and every member of ψ is an induced cycle or an induced path. The minimum cardinality of an induced graphoidal cover of G is called the induced graphoidal covering number of G and is denoted by $\eta_i(G)$ or η_i . Here we find induced graphoidal cover for some classes of graphs.

Keywords—Graphoidal cover, Induced graphoidal cover, Induced graphoidal covering number.

I. INTRODUCTION

A graph is a pair $G = (V, E)$, where V is the set of vertices and E is the set of edges. Here, we consider only nontrivial, finite, connected and simple graphs. The order and size of G are denoted by p and q respectively. The concept of graphoidal cover was introduced by B.D. Acharya and E. Sampathkumar [1] and the concept of semigraphoidal cover was introduced by S. Arumugam *et.al.* [3] (see also induced graphoidal cover by S. Arumugam [4]). The study of this parameter was initiated by S. Arumugam [4]. The reader may refer [3], [5] and [6] for the terms not defined here.

Definition I.1. [1] A graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions:

- (i) Every path in ψ has at least two vertices.
- (ii) Every vertex of G is an internal vertex of at most one path in ψ .
- (iii) Every edge of G is in exactly one path in ψ .

The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by $\eta(G)$.

Definition I.2. [2] A simple graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions:

- (i) Every path in ψ has at least two vertices.
- (ii) Every vertex of G is an internal vertex of at most one path in ψ .
- (iii) Every edge of G is in exactly one path in ψ and any two paths in ψ have at most one vertex in common.

The minimum cardinality of a simple graphoidal cover of G is called the simple graphoidal covering number of G and is denoted by $\eta_s(G)$ or η_s .

Definition I.3. [3] Let ψ be merely a partition of the edge-set $E(G)$ of G , each part of which, as an edge-induced

subgraph of G , spans either a path or a cycle which is called semigraphoidal cover of G .

Definition I.4. [4] An induced graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions:

- (i) Every path in ψ has at least two vertices.
- (ii) Every vertex of G is an internal vertex of at most one path in ψ .
- (iii) Every edge of G is in exactly one path in ψ .
- (iv) Every member of ψ is an induced cycle or an induced path.

The minimum cardinality of an induced graphoidal cover of G is called the induced graphoidal covering number of G and is denoted by $\eta_i(G)$ or η_i .

Remark I.5. We observe that every path(cycle) in a simple graphoidal cover is an induced path(cycle) but every induced graphoidal cover is not necessarily a simple graphoidal cover of a graph.

Definition I.6. Let ψ be a collection of internally edge disjoint paths in G . A vertex of G is said to be an internal vertex of ψ if it is an internal vertex of some path in ψ , otherwise it is called an external vertex of ψ .

II. MAIN RESULTS

The following result for graphoidal covering number also holds for induced graphoidal covering number.

Theorem II.1. ([3], Theorem 3.6) For any induced graphoidal cover ψ of a (p, q) - graph G , let t_ψ denote the number of external vertices of ψ and let $t = \min t_\psi$, where the minimum is taken over all induced graphoidal covers ψ of G then $\eta_i(G) = q - p + t$.

Corollary II.2. For any graph G , $\eta_i(G) \geq q - p$. Moreover, the following are equivalent

- (i) $\eta_i(G) = q - p$.
- (ii) There exists an induced graphoidal cover of G without external vertices.
- (iii) There exists a set Q of internally disjoint and edge disjoint induced graphoidal cycle or path without exterior vertices (From such a set Q of paths the required induced graphoidal cover can be obtained by adding the edges which are not covered by the paths in Q).

Corollary II.3. If there exists an induced graphoidal cover ψ of a graph G such that every vertex of G with degree at least two is internal to ψ , then ψ is a minimum induced graphoidal cover of G and $\eta_i(G) = q - p + n$, where n is the number of pendant vertices of G .

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Corollary II.4. Since every graphoidal cover of a tree T is also an induced graphoidal cover of T , we have $\eta_i(T) = n - 1$, where n is the number of pendant vertices of T .

Theorem II.5. Let G be a complete bipartite graph $K_{m,n}$, then

(i) $\eta_i(K_{1,n}) = n - 1, \quad n \geq 2$

(ii) $\eta_i(K_{2,n}) = \begin{cases} q - p + 1 & \text{if } n = 2, 3 \\ q - p & \text{if } n \geq 4 \end{cases}$

(iii) $\eta_i(K_{3,n}) = \begin{cases} q - p + 1 & \text{if } n = 3 \\ q - p & \text{if } n \geq 4 \end{cases}$

(iv) $\eta_i(K_{m,n}) = q - p, \quad n \geq m \text{ and } n \geq 4$

Proof: Let $X = \{v_1, v_2, v_3, \dots, v_m\}$ and $Y = \{w_1, w_2, w_3, \dots, w_n\}$ be a bipartition of $K_{m,n}$.

(i). When $n \geq 2$. $K_{1,n}$ is a tree with n pendant vertices and hence $\eta_i(K_{1,n}) = n - 1$.

(ii). Case(a). When $n = 2$, i.e. $K_{2,2} = C_4$, we have $\eta_i(K_{2,2}) = 1$.

Case(b). When $n = 3$, let $X = \{v_1, v_2\}$ and $Y = \{w_1, w_2, w_3\}$ be a bipartition of $K_{2,3}$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1)$ and $P_2 = (v_1, w_3, v_2)$ then $\psi = \{P_1, P_2\}$ is an induced graphoidal cover of $K_{2,3}$ with v_1 as its external vertex. Hence, $\eta_i(K_{2,3}) = q - p + 1$.

Case(c). When $n \geq 4$, let $X = \{v_1, v_2\}$ and $Y = \{w_1, w_2, w_3, \dots, w_n\}$ be a bipartition of $K_{2,n}$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1)$, $P_2 = (v_2, w_3, v_1, w_4, v_2)$ and $P_{i-2} = (v_1, w_i, v_2), i = 5, 6, \dots, n$. Then $\psi = \{P_1, P_2, P_{i-2} | i = 5, 6, \dots, n\}$ is an induced graphoidal cover of $K_{2,n}$ and every vertex is an internal vertex of exactly one path in ψ . Hence, $\eta_i(K_{2,n}) = q - p$.

(iii). Case(a). When $n = 3$, let $X = \{v_1, v_2, v_3\}$ and $Y = \{w_1, w_2, w_3\}$ be a bipartition of $K_{3,3}$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1)$, $P_2 = (v_1, w_3, v_2)$, $P_3 = (w_1, v_3, w_2)$, $P_4 = (v_3, w_3)$. Then $\psi = \{P_1, P_2, P_3, P_4\}$ is an induced graphoidal cover of $K_{3,3}$ with v_1 as its external vertex. Hence, $\eta_i(K_{3,3}) = q - p + 1$.

Case(b). When $n \geq 4$, let $X = \{v_1, v_2, v_3\}$ and $Y = \{w_1, w_2, w_3, \dots, w_n\}$ be a bipartition of $K_{3,n}$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1)$, $P_2 = (v_2, w_3, v_1, w_4, v_2)$, $P_{i-2} = (v_1, w_i, v_2), i = 5, 6, \dots, n$ and $Q = (w_1, v_3, w_2)$.

Then $\psi = \{P_1, P_2, P_3, P_4, \dots, P_{n-2}, Q\} \cup S$, where S is the set of edges not covered by $P_1, P_2, P_3, \dots, P_{n-2}, Q$ is an induced graphoidal cover of $K_{3,n}$ and every vertex is an internal vertex of a path in ψ . Hence, $\eta_i(K_{3,n}) = q - p$.

(iv). When $m \geq n$ and $n \geq 4$, let $X = \{v_1, v_2, v_3, \dots, v_m\}$ and $Y = \{w_1, w_2, w_3, \dots, w_n\}$ be a bipartition of $K_{m,n}$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1)$, $P_2 = (v_2, w_3, v_1, w_4, v_2)$, $P_{i-2} = (v_1, w_i, v_2), i = 5, 6, \dots, n$ and $Q_{j-2} = (w_1, v_j, w_2), j = 3, 4, \dots, m$.

Then $\psi = \{P_1, P_2, P_3, \dots, P_{n-2}, Q_1, Q_2, \dots, Q_{m-2}\} \cup S$, where S is the set of edges not covered by $P_1, P_2, \dots, P_{n-2}, Q_1, Q_2, \dots, Q_{m-2}$ is an induced graphoidal cover of $K_{m,n}$ and every vertex is an internal vertex of a path in ψ . Hence, $\eta_i(K_{m,n}) = q - p$. ■

Theorem II.6. If G is a unicyclic graph with n pendant

vertices and the unique cycle C_k , and j denote the number of vertices of degree greater than 2 on C_k then

$$\eta_i(G) = \begin{cases} 1 & \text{if } j = 0; \\ n + 1 & \text{if } j = 1 \text{ and the unique vertex is of deg } 3; \\ & \text{or } j = 2 \text{ and the two vertices of deg } 3 \text{ are} \\ & \text{adjacent on } C_k; \\ n & \text{otherwise.} \end{cases}$$

Proof: Let $C_k = \{v_1, v_2, v_3, \dots, v_k, v_1\}$ be the unique cycle in G .

Case(a). When $j = 0$, then $G = C_k$ so that $\eta_i(G) = 1$.

Case(b). When $j = 1$ and the unique vertex, say u , of degree 3 on C_k . Let $T = G - C_k$ be the tree rooted at u , then T has $n + 1$ pendant vertices so that $\eta_i(T) = n$. Let ψ_1 be a minimum induced graphoidal cover of T and so $|\psi_1| = n$. Then $\psi = \psi_1 \cup C_k$, where an arbitrary vertex of C_k is taken as an external vertex of C_k , is an induced graphoidal cover of G and so $|\psi| = n + 1$. Hence, $\eta_i(G) \leq n + 1$. Further, for any induced graphoidal cover ψ of G , the n pendant vertices of G and at least one vertex on C_k are external vertices in ψ so that $t \geq n + 1$. Hence, $\eta_i(G) = q - p + t \geq q - p + n + 1 = n + 1$.

When $j = 2$ and v_1 and v_2 are adjacent vertices in C_k with $\deg v_1, \deg v_2 = 3$. Let T_1 be the tree rooted at v_1 , then T_1 has $n_1 + 1$ pendant vertices so that $\eta_i(T_1) = n_1$. Let ψ_1 be a minimum induced graphoidal cover of T_1 . Similarly, T_2 is also a tree rooted at v_2 , then T_2 has $n_2 + 1$ pendant vertices so that $n = n_1 + n_2$ and $\eta_i(T_2) = n_2$. Let ψ_2 be a minimum induced graphoidal cover of T_2 . Then $\psi = \psi_1 \cup \psi_2 \cup C_k$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + |\psi_2| + 1 = n + 1$. Hence, $\eta_i(G) \leq n + 1$. Again, for any induced graphoidal cover ψ of G , the n pendant vertices of G and at least one vertex on C_k are external vertices in ψ so that $t \geq n + 1$. Hence $\eta_i(G) = q - p + t \geq q - p + n + 1 = n + 1$.

Case(c). When $j = 1$ and the unique vertex u is of $\deg u > 3$. Let T be the subgraph of G obtained by deleting all the vertices other than u of the unique cycle C_k in G so that $\eta_i(T) = n - 1$, with ψ_1 as a minimum induced graphoidal cover. Then $\psi = \psi_1 \cup C_k$, where u is taken as an external vertex of C_k , is an induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence $\eta_i(G) = n$.

If $j = 2$ and the vertices of $\deg \geq 4$ are adjacent then the result follows from case(c) with $j = 1$.

When $j = 2$ and v_1, v_3 are the two non adjacent vertices with each of $\deg v_1, \deg v_3 \geq 3$. Let $P = (v_1, v_k, v_{k-1}, \dots, v_4, v_3)$ be an induced path of length $k - 2$. Then $T = G - P$ is a tree with n pendant vertices so that $\eta_i(T) = n - 1$, with ψ_1 as a minimum induced graphoidal cover. Then $\psi = \psi_1 \cup P$ is an induced graphoidal cover of G such that every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_i(G) = n$.

When $j \geq 3$ and $\deg v_i \geq 3, i = 1, 2, \dots, r$ and $r \leq k$, where $v_i, i = 1, 2, 3, \dots, k$ are the vertices on C_k .

Suppose $k = 3$, Let $G_1 = G - (v_1 v_2)$ be a tree with n pendant vertices. Also, let T be the induced subgraph of G_1 containing v_2 and v_3 such that v_3 is a pendant vertex in T . Then T has $n_1 + 1$ pendant vertices so that $\eta_i(T) = n_1$. Let ψ_1 be a

minimum induced graphoidal cover of T and so $|\psi_1| = n_1$. Again, $T' = G_1 - T$ is also a tree with n_2 pendant vertices so that $\eta_i(T') = n_2 - 1$, where $n_1 + n_2 = n$. Let ψ_2 be a minimum induced graphoidal cover of T' and so $|\psi_2| = n_2 - 1$. Then $\psi = \psi_1 \cup \psi_2 \cup (v_1 v_2)$, is an induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_i(G) = n_1 + n_2 - 1 + 1 = n$.

Next, suppose $k > 3$. let G' be the induced subgraph of G with vertex set $\{v_1, v_2, \dots, v_s\}$, where $s < r$ and $\deg v_1, v_s \geq 3$, on the cycle C_k . Then G' has n_1 pendant vertices so that $\eta_i(G') = n_1 - 1$. Let ψ_1 be the minimum induced graphoidal cover of G' . Then $T = G - G'$ is a tree with $n_2 + 2$ pendant vertices so that $n = n_1 + n_2$ and $\eta_i(T) = n_2 + 1$. Let ψ_2 be the minimum induced graphoidal cover of T . Then $\psi = \psi_1 \cup \psi_2$, is an induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_i(G) = n$. ■

Theorem II.7. Let G be a bicyclic graph with n pendant vertices containing a $U(l; m)$ [see, [6]] and j be the number of vertices of degree greater than 2 in $U(l; m)$. Then

$$\eta_i(G) = \begin{cases} 2 & \text{if } G = U(l; m); \\ n + 2 & \text{if either } j = 1 \text{ and } \deg u_0 = 5; \text{ or } j = 2, \\ & \text{deg } u_0 = 4 \text{ and the vertex of deg 3 in} \\ & U(l; m) \text{ is adjacent to } u_0; \text{ or } j = 3, \\ & \text{deg } u_0 = 4 \text{ such that } u_0 \text{ is adjacent to} \\ & \text{both vertices } v \text{ and } w \text{ of degree 3 in } C_l \\ & \text{and } C_m \text{ respectively;} \\ n + 1 & \text{otherwise.} \end{cases}$$

Proof: Let the l -cycle be $C_l = \{u_0, u_1, \dots, u_{l-1}, u_0\}$ and the m -cycle be $C_m = \{u_0, u_l, u_{l+1}, \dots, u_{m+l-2}, u_0\}$ in G .

Case (a). Suppose $G = U(l; m)$. Then $\psi = \{C_l, C_m\}$ is an induced graphoidal cover of G with u_0 as the external vertex for both C_l and C_m . Hence, $\eta_i(G) = q - p + 1 = 2$.

Case(b). When $j = 1$ and $\deg u_0 = 5$. Then $G_1 = G - C_m$ is a unicyclic graph with one vertex of $\deg 3$ in the cycle so that $\eta_i(G_1) = n + 1$. Let ψ_1 be a minimum induced graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_m$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta_i(G) \leq n + 2$. Again, for any induced graphoidal cover ψ of G , the n pendant vertices of G and at least one vertex in $U(l; m)$ are external vertices in ψ so that $t \geq n + 1$. Hence, $\eta_i(G) = q - p + t \geq n + 1 + 1 = n + 2$.

When $j = 2$, $\deg u_0 = 4$ and $\deg v = 3$ are adjacent to u_0 in C_m . Then $G_1 = G - C_l$ is a unicyclic graph with $\deg v = 3$ so that $\eta_i(G_1) = n + 1$. Let ψ_1 be a minimum induced graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_l$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta_i(G) \leq n + 2$. Again, for any induced graphoidal cover ψ of G , the n pendant vertices of G and at least one vertex in $U(l; m)$ are external vertices in ψ so that $t \geq n + 1$. Hence, $\eta_i(G) = q - p + t \geq n + 1 + 1 = n + 2$.

When $j = 3$, $\deg u_0 = 4$ such that u_0 is adjacent to both vertices v and w of degree 3 in C_l and C_m respectively. Then $G - U(l; m)$ gives two trees, say T_1 and T_2 , rooted at v and

w with $n_1 + 1$ and $n_2 + 1$ pendant vertices respectively such that $n_1 + n_2 = n$. Also, $\eta_i(T_1) = n_1$ and $\eta_i(T_2) = n_2$. Then $\psi = \psi_1 \cup \psi_2 \cup U(l; m)$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + |\psi_2| + 2 = n_1 + n_2 + 2 = n + 2$. Hence, $\eta_i(G) \leq n + 2$. Again, for any induced graphoidal cover ψ of G , the n pendant vertices of G and at least one vertex on $U(l; m)$ are external vertices in ψ so that $t \geq n + 1$. Hence $\eta_i(G) = q - p + t \geq q - p + n + 1 = n + 2$.

Case(c). When $j = 1$ and $\deg u_0 \geq 6$. Then $G_1 = G - C_l$ is a unicyclic graph with n pendant vertices so that $\eta_i(G_1) = n$, with ψ_1 as a minimum induced graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_l$ is a minimum induced graphoidal cover of G such that every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_i(G) = n + 1$.

Similarly, we can prove for $j \geq 2$ and $\deg u_0 \geq 6$ in one cycle.

When $j = 2$ and $\deg v \geq 3$ such that u_0 and v are not adjacent. Let T denote this induced $v - u_0$ subgraph section with $n_1 + 2$ pendant vertices so that $\eta_i(T) = n_1 + 1$. Let ψ_1 be a minimum induced graphoidal cover of T . Also, $G_1 = G - T$ is a unicyclic graph with n_2 pendant vertices such that $\eta_i(G_1) = n_2$, where $n_1 + n_2 = n$. Let ψ_2 be a minimum induced graphoidal cover of G_1 . Then $\psi = \psi_1 \cup \psi_2$ is a minimum induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_i(G) = n + 1$.

Similarly, we can prove the result for $j = 3$, $\deg u_0 \geq 4$ such that u_0 is non adjacent to both v and w of degree 3 in C_l and C_m respectively.

Otherwise, let T be the induced subgraph of G with vertex set $\{u_0, u_1, \dots, u_s\}$, where $s < l$ and $\deg u_s \geq 3$, on the cycle C_l . Then T has n_1 pendant vertices so that $\eta_i(T) = n_1 - 1$. Let ψ_1 be the minimum induced graphoidal cover of T . Then $G_1 = G - T$ is a unicyclic with $n_2 + 2$ pendant vertices so that $n = n_1 + n_2$ and $\eta_i(G_1) = n_2 + 2$. Let ψ_2 be the minimum induced graphoidal cover of G_1 . Then $\psi = \psi_1 \cup \psi_2$, is an induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_i(G) = n + 1$. ■

Theorem II.8. Let G be a bicyclic graph with n pendant vertices containing a $D(l, m; i)$ [see, [6]] and j be the number of vertices of degree greater than 2 on cycles in $D(l, m; i)$. Then

$$\eta_i(G) = \begin{cases} 3 & \text{if } G = D(l, m; i); \\ n + 3 & \text{if } j = 3, \text{ exactly one vertex which is} \\ & \text{adjacent to either } u_{l-1} \text{ or } u_{l+i-1} \text{ in} \\ & D(l, m; i) \text{ is of degree 3 and } \deg u_{l-1}, \\ & u_{l+i-1} = 3; \text{ or } j = 4, v \text{ and } w \text{ are two} \\ & \text{vertices of degree 3 adjacent to } u_{l-1} \\ & \text{and } u_{l+i-1} \text{ respectively and} \\ & \text{deg } u_{l-1}, u_{l+i-1} = 3; \\ n + 2 & \text{if } \deg u_{l-1} \geq 4, \text{ deg } u_{l+i-1} = 3 \\ & \text{and } j \geq 2 \text{ in } C_l; \\ n + 1 & \text{otherwise.} \end{cases}$$

Proof: G is bicyclic, so G contains at least $C_l = \{u_0, u_1, \dots, u_{l-1}, u_0\}$, $P_i = \{u_{l-1}, u_l, \dots, u_{l+i-1}\}$ and $C_m = \{u_{l+i-1}, u_{l+i}, \dots, u_{l+m+i-2}, u_{l+i-1}\}$ in G .

Case(a). $G = D(l, m; i)$. Then $\psi = \{C_l, P_i, C_m\}$ is an induced graphoidal cover of G with u_{l-1} and u_{l+i-1} as its only external vertices. Hence, $\eta_i(G) = q - p + 2 = 3$.

Case(b). When $j = 3$ and exactly one vertex, say v , which is adjacent to either u_{l-1} or u_{l+i-1} in $D(l, m; i)$ is of degree 3. Suppose v lies in C_l . Then $G_1 = G - C_m$ is a unicyclic graph with $n+1$ pendant vertices so that $\eta_i(G_1) = n+2$. Let ψ_1 be a minimum induced graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_m$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n+3$. Hence, $\eta_i(G) \leq n+3$. Again, for any induced graphoidal cover ψ of G , the n pendant vertices of G and at least two vertices in $D(l, m; i)$ are external vertices in ψ so that $t \geq n+2$. Hence, $\eta_i(G) = q - p + t \geq n+2+1 = n+3$.

When $j = 4$, v and w are two vertices of degree 3 adjacent to u_{l-1} and u_{l+i-1} respectively and $\deg u_{l-1}, u_{l+i-1} = 3$. Let T_1 be the tree rooted at v , then T_1 has n_1+1 pendant vertices so that $\eta_i(T_1) = n_1$. Let ψ_1 be a minimum induced graphoidal cover of T_1 . Similarly, T_2 is also a tree rooted at w , then T_2 has n_2+1 pendant vertices so that $n = n_1 + n_2$ and $\eta_i(T_2) = n_2$. Let ψ_2 be a minimum induced graphoidal cover of T_2 . Then $\psi = \psi_1 \cup \psi_2 \cup D(l, m; i)$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + |\psi_2| + 3 = n_1 + n_2 + 3 = n+3$. Hence, $\eta_i(G) \leq n+3$. Again, for any induced graphoidal cover ψ of G , the n pendant vertices of G and at least two vertices on $D(l, m; i)$ are external vertices in ψ so that $t \geq n+2$. Hence $\eta_i(G) = q - p + t \geq q - p + n + 2 = n+3$.

Case(c). When $\deg u_{l-1} \geq 4$, $\deg u_{l+i-1} = 3$ and $j \geq 2$ in C_l . Here, $G_1 = G - C_m$ is a unicyclic graph with $n+1$ pendant vertices so that $\eta_i(G_1) = n+1$, with ψ_1 as a minimum induced graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_m$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n+2$. Hence, $\eta_i(G) \leq n+2$. Again, for any induced graphoidal cover ψ of G , the n pendant vertices of G and at least one vertex in $D(l, m; i)$ are external vertices in ψ so that $t \geq n+1$. Hence, $\eta_i(G) = q - p + t \geq 1 + n + 1 = n+2$.

Similarly, we can prove the result for $j = 3$, $\deg u_{l-1} = \deg u_{l+i-1} = 3$ and the vertex v of $\deg 3$ is adjacent to neither u_{l-1} nor u_{l+i-1} and $j = 4$, $\deg v, w = 3$ and at least either v is not adjacent to u_{l-1} or w is not adjacent to u_{l+i-1} .

Case(d). When $j = 2$. Then $G_1 = G - C_l$ is a unicyclic graph with n pendant vertices so that $\eta_i(G_1) = n$, with ψ_1 as a minimum induced graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_l$ is a minimum induced graphoidal cover of G such that every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_i(G) = n+1$.

Similarly, it can be proved for $j = 3$ and exactly one vertex, say v , which is adjacent to either u_{l-1} or u_{l+i-1} in $D(l, m; i)$, is of $\deg > 3$.

When $j = 3$. let v be the vertex in $D(l, m; i)$ which is non adjacent to either u_{l-1} or u_{l+i-1} in $D(l, m; i)$ is of $\deg \geq 3$ and suppose v lie in C_l . Let T denote this induced $v - u_{l-1}$ subgraph section with n_1+2 pendant vertices so that $\eta_i(T) = n_1+1$. Let ψ_1 be a minimum induced graphoidal cover of T . Also, $G_1 = G - T$ is a unicyclic graph with n_2 pendant vertices such that $\eta_i(G_1) = n_2$, where $n_1+n_2 = n$. Let ψ_2 be

a minimum induced graphoidal cover of G_1 . Then $\psi = \psi_1 \cup \psi_2$ is a minimum induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_i(G) = n+1$.

Otherwise, take an induced tree T containing all vertices in an arc of a cycle in $D(l, m; i)$ such that its end vertices are of $\deg \geq 3$. Then T has n_1 pendant vertices so that $\eta_i(T) = n_1-1$. Let ψ_1 be a minimum induced graphoidal cover of T . Also, $G_1 = G - T$ is a unicyclic graph with n_2+2 pendant vertices such that $\eta_i(G_1) = n_2+2$, where $n_1+n_2 = n$. Let ψ_2 be a minimum induced graphoidal cover of G_1 . Then $\psi = \psi_1 \cup \psi_2$ is a minimum induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_i(G) = n+1$. ■

Theorem II.9. Let G be a bicyclic graph with n pendant vertices containing a $C_m(i; l)$ [see, [6]] and j be the number of vertices of degree greater than 2 on cycles in $C_m(i; l)$. Then

$$\eta_i(G) = \begin{cases} 3 & \text{if } G = C_m(i; l) \text{ and } l = 1; \\ 2 & \text{if } G = C_m(i; l) \text{ and } l \geq 2; \\ n+3 & \text{if } j = 2, l = 1 \text{ and either } \deg u_0 \text{ or } \\ & \deg u_i = 4 \text{ in } C_m(i; l); \text{ or } j = 2, l = 1 \\ & \text{and each adjacent vertices are of } \deg 4; \\ n+2 & \text{if } j = 2, l = 1 \text{ and either } \deg u_0 \text{ or } \\ & \deg u_i \geq 4 \text{ in } C_m(i; l); \text{ or } j = 3, l = 1 \\ & \text{deg of one of } u_i \text{ and } u_0 \text{ is } 3 \text{ and the other} \\ & \text{is } 4 \text{ and the third vertex of } \deg \geq 3 \text{ is} \\ & \text{adjacent to either } u_0 \text{ or } u_i; \text{ or } j = 2, \\ & l \geq 2 \text{ and either } \deg u_0 \text{ or } \deg u_i = 4 \text{ in} \\ & C_m(i; l); \text{ or } j = 3, l \geq 2 \text{ and third vertex} \\ & \text{of } \deg 3 \text{ is adjacent to either } \deg u_0 \text{ or} \\ & \deg u_i = 4 \text{ in } C_m(i; l) \text{ is of degree } 3; \\ n+1 & \text{otherwise.} \end{cases}$$

Proof: G is bicyclic, so it contains at least $C_m = \{u_0, u_1, \dots, u_i, u_{m-1}, u_0\}$ with $m \geq 4$ the chord $P_l = \{u_0, u_m, u_{m+1}, \dots, u_{l+m-2}, u_i\}$, $l \geq 1$ and $2 \leq i \leq m-2$.

Case(a). When $G = C_m(i; l)$ and $l = 1$. Let $P_1 = (u_1, u_0, u_{m-1})$, $P_2 = (u_0, u_m, u_{m+1}, \dots, u_{l+m-2}, u_i)$, $2 \leq i \leq m-2$, and $P_3 = (u_1, u_2, u_i, \dots, u_{m-1})$ be induced paths in $C_m(i; l)$. Then $\psi = \{P_1, P_2, P_3\}$ is a minimum induced graphoidal cover of G such that any two vertices of $\deg 2$ can be taken as an external vertex. Hence, $\eta_i(G) = q - p + 2 = 3$.

Case(b). When $G = C_m(i; l)$ and $l \geq 2$. Let $P_1 = (u_0, u_1, \dots, u_{m-1}, u_0)$, $m \geq 4$ and $P_2 = (u_0, u_m, u_{m+1}, \dots, u_{l+m-2}, u_i)$, $l \geq 2$ be induced paths in $C_m(i; l)$. Then $\psi = \{P_1, P_2\}$ is a minimum induced graphoidal cover of G such that any one vertex in C_m can be taken as an external vertex. Hence, $\eta_i(G) = q - p + 1 = 2$.

Case(c). When $j = 2$, $l = 1$ and either $\deg u_0$ or $\deg u_i = 4$ in $C_m(i; l)$. Let u_s , $0 < s < i$, be the vertex in $C_m(i; l)$. Then $P_1 = (u_0, u_s)$, $P_2 = (u_s, u_1, \dots, u_i)$ be induced paths in $C_m(i; l)$. Let $G_1 = G - \{P_1, P_2\}$ is a unicyclic graph with n pendant vertices so that $\eta_i(G_1) = n+1$. Let ψ_1 be a minimum induced graphoidal cover of G_1 . Then $\psi = \psi_1 \cup \{P_1, P_2\}$ is an

induced graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 1 + 2$. Hence, $\eta_i(G) \leq n + 3$. Again, for any induced graphoidal cover ψ of G , the n pendant vertices of G and at least two vertices in $C_m(i; l)$ are external vertices in ψ so that $t \geq n + 3$. Hence, $\eta_i(G) = q - p + t \geq n + 1 + 2 = n + 3$.

Similarly, we can prove for $j = 2, l = 1$ and each adjacent vertices are of deg 4.

Case(d). When $j = 2, l = 1$ and either $deg u_0$ or $deg u_i \geq 4$ in $C_m(i; l)$. Let $u_s, 0 < s < i$, be the vertex in $C_m(i; l)$. Then $P_1 = (u_0, u_s), P_2 = (u_s, u_1, \dots, u_i)$ be induced paths in $C_m(i; l)$. Let $G_1 = G - \{P_1, P_2\}$ is a unicyclic graph with n pendant vertices so that $\eta_i(G_1) = n$. Let ψ_1 be a minimum induced graphoidal cover of G_1 . Then $\psi = \psi_1 \cup \{P_1, P_2\}$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 2$. Hence, $\eta_i(G) \leq n + 2$. Again, for any induced graphoidal cover ψ of G , the n pendant vertices of G and at least one vertex in $C_m(i; l)$ are external vertices in ψ so that $t \geq n + 2$. Hence, $\eta_i(G) = q - p + t \geq n + 2 = n + 2$.

Similarly, we can prove for $j = 3, l = 1$ degree of one of u_i and u_0 is 3 and the other is 4 and the third vertex of $deg \geq 3$ is adjacent to either u_0 or u_i .

When $j = 2, l \geq 2$ and either $deg u_0$ or $deg u_i = 4$ in $C_m(i; l)$. Let $P = (u_0 u_m u_{m+1} \dots u_{l+m-2} u_i), 2 \leq i \leq m - 2$, be the chord in $C_m(i; l)$ such that $\eta_i(P) = 1$. Then $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta_i(G_1) = n + 1$. Let ψ_1 be a minimum induced graphoidal cover of G_1 . Then $\psi = \psi_1 \cup P$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta_i(G) \leq n + 2$. Again, for any induced graphoidal cover ψ of G , the n pendant vertices of G and at least one vertex in $C_m(i; l)$ are external vertices in ψ so that $t \geq n + 2$. Hence, $\eta_i(G) = q - p + t \geq n + 2 = n + 2$.

Similarly, we can prove for $j = 3, l \geq 2$ and the third vertex of degree 3 adjacent to either $deg u_0$ or $deg u_i = 4$ in $C_m(i; l)$.

Case(e). When $j \geq 4, l = 1, u_s$ and v be the two vertices of $deg \geq 3$ in $C_m(i; 1)$ other than u_0, u_i . Let G_1 be the subgraph of G with vertex set $\{u_0, u_1, \dots, u_s, u_i\}$, where $0 < s < i$ in $C_m(i; l)$ such that both u_0 and u_i are pendent vertices. Then $T = G_1 - (u_0, u_s)$ is a tree with $n_1 + 1$ pendant vertices so that $\eta_i(T) = n_1$. Let ψ_1 be a minimum induced graphoidal cover of T . Also, $G_2 = G - G_1$ is a unicyclic graph with n_2 pendant vertices so that $n_1 + n_2 = n$ and $\eta_i(G_2) = n_2$. Let ψ_2 be a minimum induced graphoidal cover of G_1 . Then $\psi = \psi_1 \cup \psi_2 \cup (u_0, u_s)$ is an induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_i(G) = n_1 + n_2 + 1 = n + 1$.

When $j \geq 2, l \geq 2$. Let $P = (u_0 u_m u_{m+1} \dots u_{l+m-2} u_i), 2 \leq i \leq m - 2$, be the chord in $C_m(i; l)$ such that $\eta_i(P) = 1$. Then $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta_i(G_1) = n$. Let ψ_1 be a minimum induced graphoidal cover of G_1 . Then $\psi = \psi_1 \cup P$ is an induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_i(G) = n + 1$. ■

The following result for simple graphoidal covering number also holds for induced graphoidal covering number.

Theorem II.10. For the wheel $W_p = K_1 + C_{p-1}, p \geq 4$, we have

$$\eta_i(W_p) = \begin{cases} 4 & \text{if } p = 4, 5 \\ 5 & \text{if } p = 6 \\ q - p & \text{if } p \geq 7 \end{cases}$$

Theorem II.11. For the complete graph $K_p(p \geq 3)$, we have

$$\eta_i(K_p) = \begin{cases} 1 & \text{if } p = 3 \\ 4 & \text{if } p = 4 \\ q - p & \text{if } p \geq 6 \text{ and } p \text{ is even} \\ q - p + 1 & \text{if } p \geq 5 \text{ and } p \text{ is odd.} \end{cases}$$

Corollary II.12. Let C_p be a cycle with p vertices, then

$$\eta_i[(C_p)^2] = \begin{cases} q - p & \text{if } p \geq 5 \text{ and } p \text{ is even} \\ q - p + 1 & \text{if } p \geq 4 \text{ and } p \text{ is odd.} \end{cases}$$

Corollary II.13. Let C_p be a cycle with p vertices, then

$$\eta_i[T(C_p)] = \begin{cases} q - p & \text{if } p \geq 5 \text{ and } p \text{ is even} \\ q - p + 1 & \text{if } p \geq 4 \text{ and } p \text{ is odd.} \end{cases}$$

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