A modified inexact Uzawa algorithm for generalized saddle point problems

Shu-Xin Miao

Abstract—In this note, we discuss the convergence behavior of a modified inexact Uzawa algorithm for solving generalized saddle point problems, which is an extension of the result obtained in a recent paper [Z.H. Cao, Fast Uzawa algorithm for generalized saddle point problems, Appl. Numer. Math., 46 (2003) 157-171].

Keywords—Saddle point problem; Inexact Uzawa algorithm; Convergence behavior.

I. INTRODUCTION

In this note, we consider the generalized saddle point problems of the form

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}, \quad (1)$$

where $A \in R^{n\times n}$ is symmetric positive definite, $B \in R^{m\times n}$ is of full row rank, and $C \in R^{m\times m}$ is symmetric positive semidefinite, $p \in R^n$ and $q \in R^m$ are given vectors, with $m \le n$.

The generalized saddle point problems (1) arises in a wide variety of scientific and engineering applications, see [2] and references therein. Frequently, the matrices A and B are large and sparse. So iterative methods become more attractive than direct methods for solving the generalized saddle point problems (1). Many efficient iterative methods have been studied in the literature [1], [2], [8], [9], [12], [13], [15]. For example, Miao and Wang [12] studied a class of stationary iterative methods for (1) based on the work of Yun and Kim [14].

Uzawa-type algorithms are of interest because they are simple, efficient and have minimal computer memory requirements. Therefore, Uzawa-type algorithms are widely used in engineering community, especially, are used for solving saddle point problems [1], [3], [4], [5], [6], [7], [10], [11], [15].

Recently, Cao [5] consider the inexact Uzawa algorithm for solving generalized saddle point problems (1), which is an extension of the results obtained in [3]. In this note, a slight modification of the inexact Uzawa algorithm for solving generalized saddle point problems (1) (see [5]) is discussed, a bound of convergence rate is obtained.

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II. MODIFIED INEXACT UZAWA ALGORITHM

Let Q_A and Q_B be symmetric, positive definite $n \times n$ and $m \times m$ matrix, respectively, satisfying

$$\begin{aligned} & (1-\delta)(Q_A u,\ u) \le (Au,\ u) \\ < \ & (Q_A u,\ u), \ \ \forall\, u \in R^n, \\ & (1-\gamma)(Q_B v,\ v) \le ((BA^{-1}B^T + C)v,\ v) \\ \le \ & (Q_B v,\ v), \ \ \forall\, v \in R^m \end{aligned} \quad (2)$$

for some $0 < \delta < 1$ and $0 \le \gamma < 1$. Here $(\cdot,\ \cdot)$ is the Euclidean inner product in R^n or R^m. Then the inexact Uzawa algorithm for solving (1) as follows:

Algorithm 1. (INEXACT UZAWA ALGORITHM) *For* $x_0 \in R^n$ and $y_0 \in R^m$, *given, the iterative sequence* $\{(x_i,\ y_i)\}$ *is defined, for* $i = 1, 2, \cdots$, *by*

$$\begin{cases} x_{i+1} = x_i + Q_A^{-1}(p - (Ax_i + B^T y_i)), \\ y_{i+1} = y_i + Q_B^{-1}(Bx_{i+1} - Cy_i - q). \end{cases}$$

From (2), we can see that $Q_A - A$ and Q_B are symmetric and positive definite, therefore we can define an inner product in $R^n \times R^m$ by (cf. [3], [5])

$$\begin{aligned} & \left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} r \\ s \end{pmatrix} \right] \\ = \ & \left(\begin{pmatrix} Q_A - A & \\ & Q_B \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} r \\ s \end{pmatrix} \right) \\ \equiv \ & ((Q_A - A)u, r) + (Q_B v, s). \end{aligned} \quad (3)$$

The corresponding norm is denoted by

$$[|q|] = [q, q]^{1/2}, \quad \forall\, q \in R^n \times R^m. \quad (4)$$

For the inexact Uzawa algorithm 1, Cao [5] provide the following convergence theorem:

Theorem 2. Assume that (2) hold. Let x, y be the solution pair for (1), x_i, y_i be defined by the inexact Uzawa algorithm 1, and set

$$e_i = \begin{pmatrix} e_i^x \\ e_i^y \end{pmatrix} = \begin{pmatrix} x - x_i \\ y - y_i \end{pmatrix}.$$

Then for $i = 1, 2, \cdots$

$$[|e_i|] \le \rho^i [|e_0|],$$

where

$$\rho = \frac{\gamma(1-\delta) + \sqrt{\gamma^2(1-\delta^2) + 4\delta}}{2}.$$

In this note, we discuss the following slight modification of the inexact Uzawa algorithm 1 for solving generalized saddle point problem (1).

Algorithm 3. (MODIFIED INEXACT UZAWA ALGORITHM) For $x_0 \in R^n$ and $y_0 \in R^m$, *given, the iterative sequence* $\{(x_i,\ y_i)\}$ *is defined, for* $i = 1, 2, \cdots$, *by*

$$\begin{cases} x_{i+1} = x_i + Q_A^{-1}(p - (Ax_i + B^T y_i)), \\ y_{i+1} = y_i + \omega Q_B^{-1}(Bx_{i+1} - Cy_i - q), \end{cases}$$

where $\omega \in (0, 1]$ *is a real parameter.*

Remark 4. Algorithm 3 is an extension of Algorithm 1. It is also an extension of the inexact Uzawa algorithm considered in [6].

III. CONVERGENCE ANALYSIS

In what follows, we consider the convergence of the modified inexact Uzawa algorithm 2. Similar to (3), we can define inner product as

$$\left\lfloor \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} r \\ s \end{pmatrix} \right\rfloor = ((Q_A - A)u, r) + \omega^{-1}(Q_B v, s).$$

Therefore the corresponding norm for e_i can be defined as

$$|||e_i||| = [((Q_A - A)e_i^x, e_i^x) + \omega^{-1}(Q_B e_i^y, e_i^y)]^{1/2}. \tag{5}$$

We have the following convergence result for Algorithm 2.

Theorem 5. Assume that (2) hold. Then for $i = 1, 2, \cdots$

$$|||e_i||| \leq \rho_\omega^i |||e_0|||,$$

where $\rho_\omega = \max\{r_1(\omega), r_2(\omega)\}$ and

$$\begin{aligned} r_1(\omega) &= \frac{1}{2}[(1-\delta)(1-\omega(1-\gamma)) \\ &\quad + \sqrt{(1-\delta)^2(1-\omega(1-\gamma))^2 + 4\delta}\ ], \\ r_2(\omega) &= \sqrt{\delta}. \end{aligned}$$

Proof. Denote $S_a = BA^{-1}B^T + C$ and $Q_B(\omega) = \omega^{-1}Q_B$, it is easy to see that the iterative error equation can be expressed as (cf. [5])

$$\begin{pmatrix} -e_{i+1}^x \\ e_{i+1}^y \end{pmatrix} = M_1 \begin{pmatrix} e_i^x \\ e_i^y \end{pmatrix}, \tag{6}$$

where

$$M_1 = \begin{pmatrix} -(I - Q_A^{-1}A) & Q_A^{-1}B^T \\ Q_B(\omega)^{-1}B(I - Q_A A) & I - Q_B(\omega)^{-1}S_a \end{pmatrix}.$$

From [5] and (6), we know that

$$|||e_{i+1}||| \leq \sigma(M_1)|||e_i|||,$$

where $\sigma(M_1)$ is the spectrum of matrix M_1. Since M_1 is symmetric with respect to the $\lfloor \cdot, \cdot \rfloor$ inner product, its eigenvalues are real. We shall bound the positive and negative eigenvalue of M_1 in what follows.

We first provide a bound for the positive eigenvalues of M_1. Let

$$N = \begin{pmatrix} -\delta I & \delta^{1/2}L \\ \delta^{1/2}L^* & I - L^*L - Q_B(\omega)^{-1}C \end{pmatrix},$$

where $L = (I - Q_A^{-1}A)^{-1/2}Q_A^{-1}B^T$ and $L^* = Q_B(\omega)^{-1}B(I - Q_A^{-1}A)^{1/2}$. Then the largest eigenvalue of M_1 is bounded by the largest eigenvalue of N (see [5]). Let λ be a positive eigenvalue of N with corresponding eigenvector $\{\psi_1,\ \psi_2\}$, i.e.,

$$\begin{aligned} &-\delta\psi_1 + \delta^{1/2}L\psi_2 = \lambda\psi_1, \\ &\delta^{1/2}L^*\psi_1 + (I - L^*L - Q_B(\omega)^{-1}C)\psi_2 = \lambda\psi_2. \end{aligned} \tag{7}$$

Eliminating ψ_1 gives

$$-\lambda L^*L\psi_2 = (\lambda + \delta)Q_B(\omega)^{-1}C\psi_2 + (\lambda + \delta)(\lambda - 1)\psi_2.$$

From the first equation of (7), we can see that $\psi_2 \neq 0$, and hence

$$\begin{aligned} &-\lambda(Q_B(\omega)L^*L\psi_2, \psi_2) \\ = \ &(\lambda + \delta)(C\psi_2, \psi_2) + (\lambda + \delta)(\lambda - 1)(Q_B(\omega)\psi_2, \psi_2). \end{aligned} \tag{8}$$

By the first equation of (2) and the definition of L and L^* it follows that

$$\begin{aligned} (Q_B(\omega)L^*L\psi_2, \psi_2) &= (Q_A^{-1}B^T\psi_2, B^T\psi_2) \\ &\geq (1-\delta)(BA^{-1}B^T\psi_2, \psi_2). \end{aligned}$$

Now (8) imply

$$\begin{aligned} 0 &= \lambda(Q_B(\omega)L^*L\psi_2, \psi_2) + (\lambda + \delta)(C\psi_2, \psi_2) \\ &\quad + (\lambda + \delta)(\lambda - 1)(Q_B(\omega)\psi_2, \psi_2) \\ &\geq \lambda(1-\delta)(BA^{-1}B^T\psi_2, \psi_2) + (\lambda + \delta)(C\psi_2, \psi_2) \\ &\quad + (\lambda + \delta)(\lambda - 1)(Q_B(\omega)\psi_2, \psi_2) \\ &= \lambda(1-\delta)(S_a\psi_2, \psi_2) + \delta(1 + \lambda)(C\psi_2, \psi_2) \\ &\quad + (\lambda + \delta)(\lambda - 1)(Q_B(\omega)\psi_2, \psi_2) \\ &\geq \lambda(1-\delta)(1-\gamma)(Q_B\psi_2, \psi_2) \\ &\quad + (\lambda + \delta)(\lambda - 1)(Q_B(\omega)\psi_2, \psi_2) \\ &= [\lambda(1-\delta)(1-\gamma) + \omega(\lambda + \delta)(\lambda - 1)](Q_B\psi_2, \psi_2). \end{aligned}$$

Since Q_B is symmetric positive definite and $\psi_2 \neq 0$, we get

$$\lambda(1-\delta)(1-\gamma) \leq -\omega(\lambda + \delta)(\lambda - 1),$$

From which we have

$$\lambda \leq r_1(\omega),$$

where

$$\begin{aligned} r_1(\omega) &= \frac{1}{2}[(1-\delta)(1-\omega(1-\gamma)) \\ &\quad + \sqrt{(1-\delta)^2(1-\omega(1-\gamma))^2 + 4\delta}\ ]. \end{aligned}$$

Next we estimate the negative eigenvalue of M_1, let $\lambda < 0$ be an eigenvalue of M_1 with corresponding eigenvector $\{\phi_1, \phi_2\}$, i.e.,

$$\begin{cases} -(I - Q_A^{-1}A)\phi_1 + Q_A^{-1}B^T\phi_2 = \lambda\phi_1, \\ Q_B(\omega)^{-1}B(I - Q_A^{-1}A)\phi_1 \\ +(I - Q_B(\omega)^{-1}BQ_A^{-1}B^T + C)\phi_2 = \lambda\phi_2. \end{cases} \tag{9}$$

From (9), we can see that $\phi_1 \neq 0$ (cf. [5]). Thus, any eigenvector of M_1 corresponding to a negative eigenvalue must have a nonzero component ϕ_1.

From (9) we have

$$((1-\lambda)I - Q_B(\omega)^{-1}C)\phi_2 = \lambda Q_B(\omega)^{-1}B\phi_1.$$

By (2) and noting $\lambda < 0$, it follows that $(1-\lambda)I - Q_B(\omega)^{-1}C$ is invertible. Thus, we get

$$\phi_2 = \lambda((1-\lambda)I - Q_B(\omega)^{-1}C)^{-1}Q_B(\omega)^{-1}B\phi_1. \quad (10)$$

Substituting (10) into the first equation in (9) and taking an inner product with $Q_A\phi_1$ gives

$$\begin{aligned} & (1+\lambda)(Q_A\phi_1, \phi_1) \quad (11) \\ = \; & (A\phi_1, \phi_1) \\ & +\lambda(((1-\lambda)I - Q_B(\omega)^{-1}C)^{-1}Q_B(\omega)^{-1}B\phi_1, B\phi_1). \end{aligned}$$

For $\phi_1 \in R^n$, we have

$$\begin{aligned} & (((1-\lambda)I - Q_B(\omega)^{-1}C)^{-1}Q_B(\omega)^{-1}B\phi_1, B\phi_1) \\ = \; & (((1-\lambda)Q_B(\omega) - C)^{-1}B\phi_1, B\phi_1) \\ = \; & \sup_{v \in R^m} \frac{(((1-\lambda)Q_B(\omega) - C)^{-1}B\phi_1, v)^2}{(((1-\lambda)Q_B(\omega) - C)^{-1}v, v)} \\ = \; & \sup_{v \in R^m} \frac{(\phi_1, B^T v)^2}{(((1-\lambda)Q_B(\omega) - C)v, v)} \\ \leq \; & \frac{1}{1-\lambda} \sup_{v \in R^m} \frac{(\phi_1, B^T v)^2}{((Q_B(\omega) - C)v, v)} \\ \leq \; & \frac{1}{1-\lambda} \sup_{v \in R^m} \frac{(A\phi_1, \phi_1)(BA^{-1}B^T v, v)}{((Q_B(\omega) - C)v, v)}. \quad (12) \end{aligned}$$

As $\omega \in (0, 1]$, the following inequality hold

$$\begin{aligned} & \sup_{v \in R^m} \frac{(A\phi_1, \phi_1)(BA^{-1}B^T v, v)}{((Q_B(\omega) - C)v, v)} \\ \leq \; & \omega \sup_{v \in R^m} \frac{(A\phi_1, \phi_1)(BA^{-1}B^T v, v)}{((Q_B - C)v, v)}. \end{aligned}$$

Then from (12) and (2), we have

$$\begin{aligned} & (((1-\lambda)I - Q_B(\omega)^{-1}C)^{-1}Q_B(\omega)^{-1}B\phi_1, B\phi_1) \\ \leq \; & \frac{\omega}{1-\lambda} \sup_{v \in R^m} \frac{(A\phi_1, \phi_1)(BA^{-1}B^T v, v)}{((Q_B - C)v, v)} \\ \leq \; & \frac{\omega}{1-\lambda}(A\phi_1, \phi_1). \end{aligned}$$

Now (11) becomes

$$(1+\lambda)(Q_A\phi_1, \phi_1) \geq (A\phi_1, \phi_1) + \frac{\lambda\omega}{1-\lambda}(A\phi_1, \phi_1).$$

Note that $\omega \leq 1$ and $\lambda < 0$, we therefore have

$$[\lambda^2 - \delta](Q_A\phi_1, \phi_1) \leq 0.$$

Q_A is symmetric positive definite, therefore we obtain the bound for the negative eigenvalue of M_1 as

$$-\sqrt{\delta} \leq \lambda < 0.$$

We complete the proof.

Remark 6. In particular, if $\omega = 1$, then Algorithm 3 becomes Algorithm 1. Therefore, we can obtain the convergence result of Algorithm 1 (Theorem 2) directly from Theorem 5.

Remark 7. We remark that $r_2(\omega) < 1$ as $0 < \delta < 1$. It is elementary to see that $r_1(\omega) < 1 - \frac{1}{2}\omega(1-\delta)(1-\gamma)$. Therefore $\rho_\omega < 1$, that is to say that the modified inexact Uzawa method (Algorithm 3) converges if (2) hold.

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REFERENCES

[1] Z.-Z. Bai, Z.-Q. Wang, On parameterized inexact Uzawa methods for generalized saddle point problems, *Linear Algebra Appl.*, 428 (2008) 2900-2932.

[2] M. Benzi, G.H. Golub, J. Liesen, Numerical solution of saddle point problems, *Acta Numer.*, 14 (2005) 1-137.

[3] J.H. Bramble, J.E. Pasciak, A.T. Vassilev, Analysis of the inexact Uzawa algorithm for saddle point problems, *SIAM J. Numer. Anal.*, 34 (1997) 1072-1092.

[4] Y. Cao, Y. Lin, Y. Wei, Nolinear Uzawa methods for solving nonsymmetric saddle point problems, *J. Appl. Math. Comput.*, 21 (2006) 1-21.

[5] Z.-H. Cao, Fast Uzawa algorithm for generalized saddle point problems, *Appl. Numer. Math.*, 46 (2003) 157-171.

[6] X.-L. Cheng, The inexact Uzawa algorithm for saddle point problem, *Appl. Math. Letters*, 13 (2000) 1-3.

[7] H.C. Elman, G.H. Golub, Inexact and preconditioned Uzawa algorithms for saddle point problems, *SIAM J. Numer. Anal.*, 31(1994) 1645-1661.

[8] B. Fischer, R. Ramage, D.J. Silvester, A.J. Wathen, Minimum residual methods for augmented systems, *BIT Numer. Math.*, 38 (1998) 527-543.

[9] G.H. Golub, X. Wu, J.-Y. Yuan, SOR-like methods for augmented systems, *BIT Numer. Math.*, 41 (2001) 71-85.

[10] Y. Lin, Y. Wei, A convergence analysis of the nonlinear Uzawa algorithm for saddle point problems, *Appl. Math. Letters*, 20 (2007) 1094-1098.

[11] Y. Lin, Y. Wei, Fast corrected Uzawa methods for solving symmetric saddle point problems, *Calcolo*, 43 (2006) 65-82.

[12] S.-X. Miao, K. Wang, On generalized stationary iterative method for solving the saddle point problems, *J. Appl. Math. Comput.*, DOI: 10.1007/s12190-009-0369-8.

[13] X. Wu, B.P.B. Silva, J.-Y. Yuan, Conjugate gradient method for rank deficient saddle point problems, *Numer. Algorithms*, 35 (2004) 139-154.

[14] J.H. Yun, S.W. Kim, Generalized stationary iterative method for solving linear systems, *Korean J. Comput. Appl. Math.*, 5 (1998) 341-349.

[15] B. Zheng, Z.-Z. Bai, X. Yang, On Semi-Convergence of Parameterized Uzawa Methods for Singular Saddle Point Problems, *Linear Algebra Appl.*, 431 (2009) 808-817.

Uniformly persistence of a predator-prey model with Holling III type functional response

Yanling Zhu

Abstract—In this paper, a predator-prey model with Holling III type functional response

$$\begin{cases} \dot{x}(t) = x(t)(r_1(t) - b_1(t)x(t)) - \dfrac{c_1(t)x^2(t)}{kx^2(t)+1}y^m(t) \\ \dot{y}(t) = y(t)(-r_2(t) - b_2(t)y(t)) + \dfrac{c_2(t)x^2(t)}{kx^2(t)+1}y^m(t) \end{cases}$$

is studied. It is interesting that the system is always uniformly persistent, which yields the existence of at least one positive periodic solutions for the corresponding periodic system. The result improves the corresponding ones in [11]. Moreover, an example is illustrated to verify the results by simulation.

Keywords—Predator-prey model, Uniformly persistence, Comparison theorem, Holling III type functional response.

I. Introduction

THE first differential equation of predator-prey model was introduced by A.J. Lotka (1925) and V. Volterra (1926), respectively. After that many more complicated but realistic predator-prey model have been formulated by ecologists and mathematicians. The dynamic relationship between predators and their preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. Recently predator-prey models with the mutual interference between the predators and preys have been extensively studied(see [1-7]), which was introduced by Hassell in 1971. From the observation Hassell introduced the concept of mutual interference constant $m(0 < m \leq 1)$ and established a Volterra model with mutual interference as follows (see [8-10])

$$\begin{cases} \dot{x} = xg(x) - \varphi(x)y^m, \\ \dot{y} = y\left(-d + k\varphi(x)y^{m-1} - q(y)\right). \end{cases}$$

In [11] the authors discussed a Lotka-Volterra model with mutual interference and Holling III type functional response as follows

$$\begin{cases} \dot{x}(t) = x(t)(r_1(t) - b_1(t)x(t)) - \dfrac{c_1(t)x^2(t)}{x^2(t)+k^2}y^m(t), \\ \dot{y}(t) = y(t)(-r_2(t) - b_2(t)y(t)) + \dfrac{c_2(t)x^2(t)}{x^2(t)+k^2}y^m(t), \end{cases} \tag{1}$$

where x is the size of the prey population, and y is the size of the predator population; $k > 0$ is a constant; r_1, b_1, r_2, b_2, c_1 and c_2 are positive functions. In [11] some sufficient conditions are obtained for the existence, uniqueness and global attractivity of positive periodic solution of the model. But the authors do not discuss the uniformly persistence of the model. As far as we know, the existence of periodic solution is the special persistence. So the discussion of the uniformly persistence of the model is very important and significative.

Motivated by the above reason, in this paper by using some new analysis techniques and comparison theorem we investigate the permanence of system (2) as follows

$$\begin{cases} \dot{x}(t) = x(t)(r_1(t) - b_1(t)x(t)) - \dfrac{c_1(t)x^2(t)}{kx^2(t)+1}y^m(t), \\ \dot{y}(t) = y(t)(-r_2(t) - b_2(t)y(t)) + \dfrac{c_2(t)x^2(t)}{kx^2(t)+1}y^m(t). \end{cases} \tag{2}$$

Obviously, the models in [1-3] are the special cases of (2) with $m = 1$, i.e., there is no mutual interference between the predator and prey. Here, we only investigate system (2) in the case of $0 < m < 1$. It is interesting that the system is always uniformly persistent.

II. Definition and Lemmas

Definition 2.1 System (2) is said to be uniformly persistent if there exist positive constants $m_i,\ M_i,\ i = 1, 2$ and $T > 0$ such that

$$m_1 \leq x(t) \leq M_1; \quad m_2 \leq y(t) \leq M_2, \quad \text{for } t \geq T,$$

for any positive solution $(x(t), y(t))$ of system (2).

Lemma 2.1 (*See [12]*) If $a > 0,\ b > 0$, and

$$z'(t) \geq (\leq) b - a\,z(t), z(0) > 0,$$

then

$$z(t) \geq (\leq) \frac{b}{a}\,[1 + (\frac{az(0)}{b} - 1)e^{-at}], \forall t \geq 0.$$

Lemma 2.2 If $a > 0,\ b > 0$, and

$$z'(t) \geq (\leq) z(t)\,(b - a\,z(t)), z(0) > 0,$$

then

$$z(t) \geq (\leq) \frac{b}{a}\,[1 + (\frac{b}{az(0)} - 1)e^{-bt}]^{-1}, \forall t \geq 0.$$

Proof. From

$$z'(t) \geq z(t)\,(b - a\,z(t)),$$

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we can easily obtain

$$\frac{d(z^{-1})}{dt} \le a - b\,z^{-1}.$$

By Lemma 2.1 we have

$$z^{-1}(t) \le \frac{a}{b}\,[1 + (\frac{bz^{-1}(0)}{a} - 1)e^{-bt}], \forall t \ge 0,$$

i.e.,

$$z(t) \ge \frac{b}{a}\,[1 + (\frac{b}{az(0)} - 1)e^{-bt}]^{-1}, \forall t \ge 0.$$

Similarly, we can prove if

$$z'(t) \le z(t)\,(b - a\,z(t)),$$

then

$$z(t) \le \frac{b}{a}\,[1 + (\frac{b}{az(0)} - 1)e^{-bt}]^{-1}, \forall t \ge 0.$$

Lemma 2.3 If $a > 0,\ b > 0$, and

$$z'(t) \ge (\le) z^m(t)\,(b - a\,z^{1-m}(t)), z(0) > 0,$$

then

$$z(t) \ge (\le)[\frac{b}{a} + (z^{1-m}(0) - \frac{b}{a})e^{-a(1-m)t}]^{\frac{1}{1-m}}, \forall t \ge 0.$$

Proof. From

$$z'(t) \ge z^m(t)\,(b - a\,z^{1-m}(t)),$$

we can easily obtain

$$\frac{d(z^{1-m})}{dt} \ge (1-m)(b - a\,z^{1-m}).$$

By Lemma 2.1 we have

$$z^{1-m}(t) \ge \frac{b}{a} + (z^{1-m}(0) - \frac{b}{a})e^{-a(1-m)t}], \forall t \ge 0,$$

i.e.,

$$z(t) \ge [\frac{b}{a} + (z^{1-m}(0) - \frac{b}{a})e^{-a(1-m)t}]^{\frac{1}{1-m}}, \forall t \ge 0.$$

Similarly, we can prove the other part of this Lemma.

Before the main results we give some useful notations as follows, for any continuous bounded function f defined on $[0, +\infty)$,

$$f^L := \inf_{t\in[0,+\infty)}\{f(t)\},\ f^U := \sup_{t\in[0,+\infty)}\{f(t)\}$$

and

$$K_1 := \frac{r_1^U}{b_1^L} + \varepsilon,\ K_2 := \frac{c_2^U}{2kr_2^L} + \varepsilon,$$

where ε is a positive constant.

III. Main results

Theorem 3.1 System (2) is uniformly persistent.

Proof. The first equation in system (2) leads to that

$$x'(t) \le x(t)\left[r_1^U - b_1^L x(t)\right], \forall t \ge 0,$$

which together with Lemma 2.2 yields that

$$x(t) \le \frac{r_1^U}{b_1^L}[1 + (\frac{r_1^U}{b_1^L x(0)} - 1)e^{-r_1^U t}]^{-1}, \forall t \ge 0. \quad (3)$$

Thus, $\forall \varepsilon > 0, \exists T_1 > 0$, such that

$$x(t) \le \frac{r_1^U}{b_1^L} + \varepsilon =: K_1, \text{for } t \ge T_1. \quad (4)$$

On the other hand, the second equation in system (2) implies

$$y'(t) \le y(t)\left[-r_2^L + \frac{c_2^U}{2k}y^{m-1}(t)\right]$$

$$\le y^m(t)\left[\frac{c_2^U}{2k} - r_2^L y^{1-m}(t)\right], \forall t \ge 0,$$

which together with Lemma 2.3 yields that $\forall t \ge 0$,

$$y(t) \le \left[\frac{c_2^U}{2kr_2^L} + \left(y^{1-m}(0) - \frac{c_2^U}{2kr_2^L}\right)e^{-r_2^L(1-m)t}\right]^{\frac{1}{1-m}}. \quad (5)$$

Therefore, for above $\varepsilon > 0, \exists T_2 > 0$, such that

$$y(t) \le \frac{c_2^U}{2kr_2^L} + \varepsilon =: K_2, \text{for } t \ge T_2. \quad (6)$$

Furthermore, from the first equation in system (2) we get

$$x'(t) \ge x(t)\left[r_1^L - b_1^U x(t) - c_1(t)x(t)y^m(t)\right]$$

$$\ge x(t)\left[r_1^L - (b_1^U + c_1^U K_2^m)x(t)\right].$$

It follows from Lemma 2.2 that there exists a constant $T_3 \in R_+$ such that

$$x(t) \ge \frac{r_1^L}{b_1^U + c_1^U K_2^m} - \varepsilon, \text{for } t \ge T_3.$$

Noticing that ε is an arbitrary small constant, we can let ε be so small that

$$\varepsilon < \frac{r_1^L}{2b_1^U + 2c_1^U K_2^m}.$$

So we get

$$x(t) \ge \frac{r_1^L}{2b_1^U + 2c_1^U K_2^m} =: K_3, \text{for } t \ge T_3. \quad (7)$$

Similarly, the second equation in system (2) yields

$$y'(t) \ge y(t)\left(-r_2^U - b_2^U K_2 + \frac{c_2^L K_3}{kK_1^2 + 1}y^{m-1}(t)\right]$$

$$= y^m(t)\left(\frac{c_2^L K_3}{kK_1^2 + 1} - (r_2^U + b_2^U K_2)y^{1-m}(t)\right].$$

it follows from Lemma 2.3 that for the above ε there exists $T_4 > 0$ such that

$$y^{1-m}(t) \ge \frac{c_2^L K_3}{(kK_1^2 + 1)(r_2^U + b_2^U K_2)} - \varepsilon, \text{for } t \ge T_4.$$

Let ε be so small that

$$\varepsilon < \frac{c_2^L K_3}{2(kK_1^2+1)(r_2^U + b_2^U K_2)},$$

so we get

$$y(t) \geq \left[\frac{c_2^L K_3}{2(kK_1^2+1)(r_2^U + b_2^U K_2)}\right]^{\frac{1}{1-m}} =: K_4, \text{for } t \geq T_4. \tag{8}$$

Let $T_0 = \max\{T_1, T_2, T_3, T_4\}$, by formula (4), (6), (7) and (8) we get

$$K_3 \leq x(t) \leq K_1 \text{ and } K_4 \leq y(t) \leq K_2, \text{ for } t > T_0.$$

Now we complete the proof of Theorem 3.1.

Remark If system (2) is a periodic system, then by Brouwer fixed point theorem we know System (2) is uniformly persistent, which implies that system (2) has at least one positive T-periodic solution. Thus the result of the existence of positive periodic solutions is improved, which is weaker than the ones in [11].

IV. Stimulation

As an application, we consider the following system:

$$\begin{cases} \dot{x}(t) = x(t)(r_1(t) - b_1(t)x(t)) - \dfrac{c_1(t)x^2(t)}{kx^2(t)+1} y^m(t), \\ \dot{y}(t) = y(t)(-r_2(t) - b_2(t)y(t)) + \dfrac{c_2(t)x^2(t)}{kx^2(t)+1} y^m(t), \end{cases} \tag{9}$$

where

$$r_1(t) = 5 - 0.3\sin t, b_1(t) = 4 + 0.5\cos t, c_1(t) = 1 + 0.6\sin t,$$

$$r_2(t) = 3 + 0.4\sin t, b_2(t) = 4 + 0.7\sin t, c_2(t) = 5 + 0.8\cos t,$$

$k = 0.5$ and $m = 2/3$.

We easily obtain system (9) is uniformly persistent.

Noticing that this system is a periodic system, system (9) has at least one positive 2π-periodic solution. In order to verify our conclusions further, we take the initial values by

$$(x(0), y(0)) = (0.3, 0.3), (x(0), y(0)) = (0.6, 0.7)$$

and

$$(x(0), y(0)) = (1.8, 0.8), (x(0), y(0)) = (2, 0.2),$$

respectively.

From the following figure, one can easily see that the positive solutions of system (9) are eventually tend to a periodic orbits, which yields that the predator and prey are uniformly persistent.

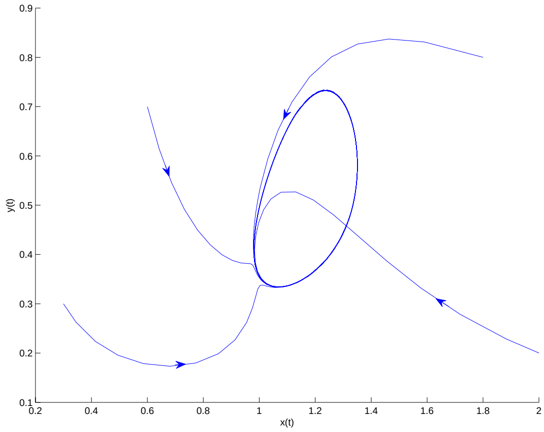

Fig. Evolution of the solutions of system (9).

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References

[1] Z. Teng, On the persistence and positive periodic solution for planar competing Lotka-Volterra systems, *Ann. of Diff. Eqs.* **13** (1997) 275-286.
[2] Z. Teng and L. Chen, Necessary and sufficient conditions for existence of positive periodic solutions of periodic predator-prey systems, *Acta. Math. Scientia* **18** (1998) 402-406(in Chinese).
[3] F. Chen, The permanence and global attractivity of Lotka-Volterra competition system with feedback controls, *Nonlinear Anal.: RWA* **7** (2006) 133-143.
[4] C. Egami, N. Hirano, Periodic solutions in a class of periodic delay predator-prey systems, *Yokohama Math. J.* **51**(2004)45-61.
[5] Z. Teng, Y. Yu, The stability of positive periodic solution for periodic predator-prey systems,*Acta. Math. Appl. Sinica* **21**(1998)589-596.(in Chinese)
[6] M. Fan, K. Wang, D. Jiang, Existence and global attractivity of positive periodic solutions of periodic n-species Lotka-Volterra competition systems with several deviating arguments, *Math. Biosci.* **160**(1999)47-61.
[7] F. Ayala, M. Gilpin and J. Etherenfeld, Competition between species: theoretical systems and experiment tests, *Theory Population Biol.* **4**(1973)331-356.
[8] M. Hassell, G. Varley, New inductive population model for insect parasites and its bearing on biological control, *Nature* **223**(5211)(1969)1133-1137.
[9] M. Hassell, Mutual Interference between Searching Insect Parasites, *J. Anim. Ecol.* 40(1971)473-486.
[10] M. Hassell, Density dependence in single-species population, *J. Anim. Ecol.* 44(1975)283-295.
[11] X. Wang, Z. Du and J. Liang, Existence and global attractivity of positive periodic solution to a Lotka–Volterra model, *Nonlinear Anal.: RWA* doi: 10.1016/j.nonrwa.2010.03.011
[12] F. Chen, On a nonlinear nonautonomous predator-prey model with diffusion and distributed delay, *J. Comput. Appl. Math.* **180** (2005) 33-49.