The ratios between the spectral norm, the numerical radius and the spectral radius

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Abstract—Recently, Uhlig [*Numer. Algorithms*, 52(3):335-353, 2009] proposed open questions about the ratios between the spectral norm, the numerical radius and the spectral radius of a square matrix. In this note, we provide some observations to answer these questions.

Keywords—Spectral norm, Numerical radius, Spectral radius, Ratios

I. Introduction

THE numerical radius $w(A)$ of an $n \times n$ matrix A is the real number

$$w(A) = \max_{z \in F(A)} |z|,$$

where $F(A)$ denotes the field of values (or numerical range) of A, defined by

$$F(A) = \{x^*Ax : x \in \mathbb{C}^n, \|x\|_2 = 1\}.$$

The spectral radius $\rho(A)$ of A is the real number

$$\rho(A) = \max_{z \in \sigma(A)} |z|,$$

where $\sigma(A)$ denotes the spectrum of A. The spectral norm of A is defined by

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2.$$

In this note, we consider the ratios

$$s(A) = \|A\|_2/w(A)$$

and

$$\tau(A) = w(A)/\rho(A).$$

It is well known that

$$0 \leq \rho(A) \leq w(A) \leq \|A\|_2 \leq 2w(A).$$

Thus,

$$1 \leq s(A) \leq 2$$

and

$$1 \leq \tau(A) \leq \infty.$$

Here we employ the convention that $\tau(A) = \infty$ for $\rho(A) = 0$. Obviously, $s(zA) = s(A)$ and $\tau(zA) = \tau(A)$ for all $z \neq 0$. It follows from $\rho(A^m) = [\rho(A)]^m$ and $w(A^m) \leq [w(A)]^m$ that $\tau(A^m) \leq [\tau(A)]^m$.

Recently, Uhlig [13] proposed the questions: What do the ratios $s(A)$ and $\tau(A)$ indicate about the matrix A? What can one conclude about A when these ratios are large or very different, or when they are nearly equal? In this note, we provide some observations to answer these questions.

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II. The ratios between the spectral norm, the numerical radius and the spectral radius

A. The extreme cases $\tau(A) = 1$, $s(A) = 1$ and $s(A) = 2$

In this subsection, we review the existing results for the extreme cases $\tau(A) = 1$, $s(A) = 1$ and $s(A) = 2$, respectively. We focus on the relation between $s(A)$ and $\tau(A)$.

A matrix A is said to be *spectral* if $w(A) = \rho(A)$, i.e., $\tau(A) = 1$. The spectral matrices have been investigated by several researchers. We have the following results (see [5] and [9, p.60]).

Proposition 1. *Let $A \in \mathbb{C}^{n\times n}$ such that $\tau(A) = 1$.*

- *If $n \leq 2$, then $s(A) = 1$, and A is a normal matrix.*
- *If $n > 2$, then A is unitarily similar to a triangle matrix of the form*

$$\begin{bmatrix} \Lambda_k & 0 \\ 0 & B \end{bmatrix}, \tag{1}$$

where $1 \leq k \leq n$,

$$\Lambda_k = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix},$$

$$B = \begin{bmatrix} \lambda_{k+1} & * & * \\ & \ddots & * \\ & & \lambda_n \end{bmatrix},$$

and

$$\rho(B) < \rho(A) = |\lambda_1| = \cdots = |\lambda_k|,$$

$$w(B) \leq \rho(A).$$

Furthermore, if $\rho(A) < \|B\|_2$,

$$1 < s(A) \leq 2;$$

otherwise, $s(A) = 1$.

A matrix A is said to be *radial* if $w(A) = \|A\|_2$, i.e., $s(A) = 1$. We have the following results (see [9, p.45]).

Proposition 2. *Let $A \in \mathbb{C}^{n\times n}$ such that $s(A) = 1$. Then $\tau(A) = 1$.*

- *If $n \leq 2$, then A is a normal matrix.*
- *If $n > 2$, then A is unitarily similar to a block diagonal matrix of the form* (1) *such that $\rho(B) < \rho(A)$ and $\|B\|_2 \leq \rho(A)$.*

Remark 3. Note that $s(A) \approx 1$ does not imply that $\tau(A) \approx 1$. See Example 4.

Example 4 (A scaled Jordan block)**.** Let

$$J_n^\alpha(\lambda) = \begin{bmatrix} \lambda & \alpha & & \\ & \lambda & \ddots & \\ & & \ddots & \alpha \\ & & & \lambda \end{bmatrix}_{n\times n} = \lambda I + N \qquad (2)$$

be a matrix of order $n > 1$. Then $F(J_n^\alpha(\lambda))$ is a disk centered at λ with radius $|\alpha|\cos\frac{\pi}{n+1}$ (see [10, Theorem 2.1]). We have $\rho(J_n^\alpha(\lambda)) = |\lambda|$, $w(N) = |\alpha|\cos\frac{\pi}{n+1}$, $w(J_n^\alpha(\lambda)) = |\lambda| + |\alpha|\cos\frac{\pi}{n+1}$ and $\|J_n^\alpha(\lambda)\|_2 \le |\lambda| + |\alpha|$. Then

$$\tau(J_n^\alpha(\lambda)) = 1 + \frac{|\alpha|}{|\lambda|}\cos\frac{\pi}{n+1},$$

$$s(J_n^\alpha(\lambda)) \le \frac{1+|\alpha/\lambda|}{1+|\alpha/\lambda|\cos\frac{\pi}{n+1}} \le \frac{1}{\cos\frac{\pi}{n+1}}.$$

Thus, when $n \to \infty$ and $|\alpha|/|\lambda| \to \infty$, $s(J_n^\alpha(\lambda)) \to 1$. However, $\tau(J_n^\alpha(\lambda)) \to \infty$.

Propositions 1 and 2 give the answers of the questions ((1)(2)) of [13, p.352]. When $s(A) = 2$, we have the following result (see [8, p.18-7]).

Proposition 5. *Let $A \in \mathbb{C}^{n\times n}$ such that $s(A) = 2$. Then A is unitarily similar to a block diagonal matrix of the form*

$$\begin{bmatrix} \|A\|_2 J_2(0) & \\ & B \end{bmatrix},$$

where $J_2(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $w(B) \le \frac{\|A\|_2}{2}$.

By Proposition 5, it is easy to show that in the case $s(A) = 2$, $1 \le \tau(A) \le \infty$.

B. Upper bounds for $s(A)$ and $\tau(A)$

Let

$$A = U\Sigma V^*, \qquad \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \qquad (3)$$

be a singular value decomposition of A, where U and V are unitary and $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n \ge 0$. Denote the 2-norm condition number of a nonsingular matrix A by

$$\kappa(A) := \|A\|_2\|A^{-1}\|_2 = \sigma_1/\sigma_n.$$

Proposition 6. *Let $A \in \mathbb{C}^{n\times n}$ be as in* (3) *such that $\sigma_n > 0$. Then $s(A) \le \kappa(A)$ and $\tau(A) \le \kappa(A)$. In particular, the following statements are equivalent:*

(i) $s(A) = \kappa(A)$.
(ii) $\tau(A) = \kappa(A)$.
(iii) $\kappa(A) = 1$.
(iv) *A is a nonzero multiple of a unitary matrix.*

Proposition 7. *Let $A \in \mathbb{C}^{n\times n}$ be as in* (3) *such that $\sigma_n > 0$ and $s(A) = \tau(A)$. Then $s(A) = \tau(A) \le \sqrt{\kappa(A)}$ and the equality holds if and only if $\kappa(A) = 1$.*

Note that $\sigma_n \le \rho(A) \le w(A) \le \|A\|_2 = \sigma_1$ and if $\rho(A) = \sigma_n > 0$ then $\sigma_1 = \cdots = \sigma_n$. The proofs of Propositions 6 and 7 are trivial.

By Proposition 6, a large $\tau(A)$ implies that A is ill conditioned and if A is well conditioned, i.e., $\kappa(A) \approx 1$, then $s(A) \approx 1$ and $\tau(A) \approx 1$. For a diagonalizable (singular or nonsingular) nonzero matrix $A = X\Lambda X^{-1}$, we have $s(A), \tau(A) \le \|A\|_2/\|\Lambda\|_2 \le \kappa(X)$. Thus, a large $\tau(A)$ implies that any eigenvector basis of the diagonalizable matrix A is ill conditioned.

Remark 8. Let $A \in \mathbb{C}^{n\times n}$ be singular. The generalized 2-norm condition number is defined by $\kappa^\dagger(A) = \sigma_1/\sigma_r$, where $r = \mathrm{rank}(A)$ is the rank of A. In general, we do not have $s(A) \le \kappa^\dagger(A)$ and $\tau(A) \le \kappa^\dagger(A)$. For example, let

$$A = \begin{bmatrix} \epsilon & 1 & 0 \\ 0 & \epsilon & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have $s(A) \to \sqrt{2}$, $\tau(A) \to \infty$ and $\kappa^\dagger(A) \to 1$ as $\epsilon \to 0$.

Let

$$A = U(\Lambda + N)U^* \qquad (4)$$

be a Schur decomposition of A, where U is a unitary matrix, Λ is a diagonal matrix whose diagonal elements are the eigenvalues of A and N is a strictly upper triangular matrix.

Proposition 9. *Let $A \in \mathbb{C}^{n\times n}$ be as in* (4)*. Then*

$$w(A) \le \rho(A) + w(N),$$

and if $\rho(A) \ne 0$,

$$\tau(A) \le 1 + w(N)/\rho(A).$$

Proof: Since U is unitary $F(A) = F(\Lambda + N)$ [9, p.11]. Then

$$\begin{aligned} w(A) &= \max_{\|x\|_2=1} |x^*(\Lambda+N)x| \\ &\le \max_{\|x\|_2=1} |x^*\Lambda x| + \max_{\|x\|_2=1} |x^*Nx| \\ &= \rho(A) + w(N). \end{aligned}$$

The proof of the second inequality is trivial. ■

The bound in Proposition 9 is attainable. For example, let $J_n^\alpha(\lambda)$ be as in (2). Assume $\lambda \ne 0$. We have

$$\tau(J_n^\alpha(\lambda)) = 1 + \frac{w(N)}{\rho(J_n^\alpha(\lambda))}.$$

Another obvious bound for $\tau(A)$ is $\tau(A) \le \|A\|_2/\rho(A)$. When $\tau(A) = \|A\|_2/\rho(A)$, i.e., $s(A) = 1$, we have $\tau(A) = 1$ (see Proposition 2).

III. A SUFFICIENT CONDITION FOR $0 \in F(A)$

For the matrix $J_n^\alpha(\lambda)$ in Example 4 with $\lambda \ne 0$, if $|\alpha|$ is sufficiently small, then $0 \notin F(J_n^\alpha(\lambda))$. And when $\tau(J_n^\alpha(\lambda)) \ge 2$, $0 \in F(J_n^\alpha(\lambda))$. So a natural question is: Does there exist a constant $c > 1$ s.t., if $\tau(A) \ge c$, then $0 \in F(A)$? The answer to this question is positive. It was proved in [2, Lemma 2.6] that for any $A \in \mathbb{C}^{n\times n}$ if 0 is not an interior point of $F(A)$, then $\tau(A) \le n$. Here we give a slightly different version. For completeness we include its proof, which is similar to that of Lemma 2.6 of [2].

Theorem 10. *Let $A \in \mathbb{C}^{n\times n}$ such that $\tau(A) \geq n$. Then $0 \in F(A)$.*

Proof: It is sufficient to prove if $0 \notin F(A)$ then $\tau(A) < n$. It is well known that if $0 \notin F(A)$ then there exists a real number θ such that the Hermitian matrix $H(e^{i\theta}A) = (e^{i\theta}A + e^{-i\theta}A^*)/2$ is positive definite; see, e.g., [9, p.21]. By rotating A, we assume that the Hermitian part $H(A) = (A + A^*)/2$ of A is positive definite. Since $F(A)$ is unitary similarity invariant [9, p.11], we also assume A is in upper triangular form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ & & & a_{nn} \end{bmatrix}.$$

The positive definiteness of $H(A)$ implies that for all $i, j = 1, \cdots, n$,

$$\frac{1}{2}\begin{bmatrix} 2\mathrm{Re}(a_{ii}) & a_{ij} \\ \overline{a_{ij}} & 2\mathrm{Re}(a_{jj}) \end{bmatrix}$$

are positive definite. Then

$$|a_{ij}| < 2\sqrt{\mathrm{Re}(a_{ii})\mathrm{Re}(a_{jj})} \leq 2\rho(A).$$

Let $|A| = (|a_{ij}|)$. By Gershgorin circle theorem, $\rho(|A| + |A|^T) < 2n\rho(A)$. Then $\tau(A) < n$ follows from $w(A) \leq w(|A|) = \rho(|A| + |A|^T)/2$ (see [9, p.44]). ∎

Remark 11 (A geometric interpretation of the 2×2 case). If $A \in \mathbb{C}^{2\times 2}$ has eigenvalues λ_1 and λ_2, then $F(A)$ is an (possibly degenerate) elliptical disk with foci λ_1 and λ_2. Since any elliptical disk can be covered by two circular disks centered at λ_1 and λ_2 with radius a, where a is the length of the semi-major axis of the elliptical disk (see Figure 1), we have $w(A) \leq \rho(A) + a$. Thus, $\tau(A) \geq 2$ means $\rho(A) \leq a$. We have $|\lambda_1| + |\lambda_2| \leq 2\rho(A) \leq 2a$. Therefore, $0 \in F(A)$.

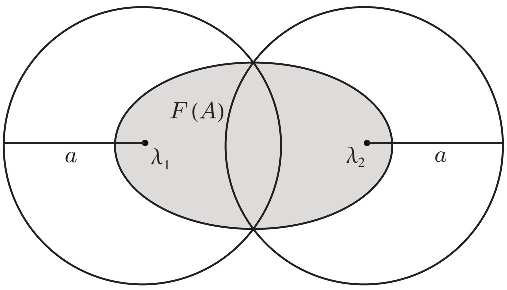

Fig. 1. Any elliptical disk can be covered by two circular disks centered at foci with radius a, where a is the length of the semi-major axis of the elliptical disk.

By Remark 11, if the numerical range of an $n \times n$ matrix A is an elliptical disk with foci $c_1, c_2 \in \sigma(A)$, the condition in Theorem 10 can be reduced to $\tau(A) \geq 2$.

Corollary 12. *Let $A \in \mathbb{C}^{n\times n}$ such that $\tau(A) \geq n$. Then the Hermitian matrix $H(e^{i\theta}A)$ is neither positive nor negative definite. Furthermore, if $\tau(A) > n$ then $H(e^{i\theta}A)$ is indefinite.*

Note that $w(A) = w(e^{i\theta}A)$ and $\rho(A) = \rho(e^{i\theta}A)$. The proof of Corollary 12 is easy. The following example shows that $H(A)$ may be semi-definite when $\tau(A) = n$.

Example 13 (An upper triangular Toeplitz matrix Z_n satisfying $\tau(Z_n) = n$)**.** Let

$$Z_n = \begin{bmatrix} 1 & 2 & \cdots & 2 \\ & 1 & \ddots & \vdots \\ & & \ddots & 2 \\ & & & 1 \end{bmatrix}_{n\times n}.$$

The matrix Z_n is given in [2] to show that the bound in Lemma 2.6 of [2] is sharp. We have $w(Z_n) = \rho(H(Z_n)) = n$, $\rho(Z_n) = 1$ and $\tau(Z_n) = n$. The origin is on the boundary of $F(Z_n)$. Obviously, $H(Z_n)$ is the matrix with all the entries being 1 and is positive semidefinite.

Let $\mathbb{P}_k$ denote the set of (complex) polynomials of degree at most k. The *polynomial numerical hull of A of degree k,*

$$\mathcal{V}^k(A) := \{z \in \mathbb{C} : |p(z)| \leq \|p(A)\|_2, \forall p \in \mathbb{P}_k\}$$

is introduced by Nevanlinna [11, p.41]. We have (see [11], [7])

$$\mathcal{V}^1(A) = F(A). \tag{5}$$

By (5), we have

$$F(A) = \bigcap_{z\in\mathbb{C}} \{\lambda \in \mathbb{C} : |\lambda - z| \leq \|A - zI\|_2\}.$$

Then $0 \in F(A)$ implies $|z| \leq \|A - zI\|_2$ for all $z \in \mathbb{C}$. We have the following corollary.

Corollary 14. *Let $A \in \mathbb{C}^{n\times n}$ such that $\tau(A) \geq n$. Then $\min_{z\in\mathbb{C}} \|I - zA\|_2 = 1$.*

Proof: Since $\tau(A) \geq n$, we have $|z| \leq \|A - zI\|_2$ for all $z \in \mathbb{C}$. Then $\|I - A/z\|_2 \geq 1$ for all $z \neq 0$. Note that $\|I - zA\|_2 = 1$ when $z = 0$. The proof is completed. ∎

IV. Concluding remarks

In this note, we discuss the existing results for the ratios $s(A)$ and $\tau(A)$. We also provide several upper bounds for them. If no further known conditions for A are given, there is no obvious relation between $s(A)$ and $\tau(A)$ except that $s(A) = 1$ implies $\tau(A) = 1$. If $\tau(A) \gg 1$, the matrix A is extremely ill conditioned and highly non-normal.

We complete this note by discussing the convergence of GMRES [12] for the linear system

$$Ax = b, \qquad A \in \mathbb{C}^{n\times n}, \qquad b \in \mathbb{C}^n. \tag{6}$$

Given an initial guess x_0 for the solution of (6), at the iteration step $m(\geq 1)$, GMRES yields the approximate solution x_m in the affine subspace $x_0 + \mathcal{K}_m(A, r_0)$ such that

$$\|r_m\|_2 := \|b - Ax_m\|_2 = \min_{y\in x_0+\mathcal{K}_m(A,r_0)} \|b - Ay\|_2,$$

where

$$\mathcal{K}_m(A, r_0) := \mathrm{span}\{r_0, Ar_0, \cdots, A^{m-1}r_0\}$$

is the mth Krylov subspace generated by the matrix A and the initial residual vector $r_0 := b - Ax_0$. For a diagonalizable matrix $A = X\Lambda X^{-1}$, one has the estimate (see, e.g., [6, p.54])

$$\frac{\|r_m\|_2}{\|r_0\|_2} \leq \kappa(X) \min_{\substack{\phi_m \in \mathbb{P}_m \\ \phi_m(0)=1}} \max_{1 \leq i \leq n} |\phi_m(\lambda_i)|. \qquad (7)$$

Obviously, this bound (7) does not always yield satisfactory results when $\tau(A) \gg 1$ due to $\kappa(X) \geq \tau(A)$. For a general matrix, one has the estimate (see [3, Corollary 6.2])

$$\frac{\|r_m\|_2}{\|r_0\|_2} \leq [1 - \nu(A)\nu(A^{-1})]^{m/2}, \qquad (8)$$

where $\nu(A) = \min\{|z| : z \in F(A)\}$ is the distance from the origin to $F(A)$. Thus, this bound (8) is useless when $\tau(A) \geq n$. The same conclusion also applies to the bound (3.3) in [4] and the bound (2.1) in [1].

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