Existence and stability analysis of discrete-time fuzzy BAM neural networks with delays and impulses

Chao Wang and Yongkun Li

Abstract—In this paper, the discrete-time fuzzy BAM neural network with delays and impulses is studied. Sufficient conditions are obtained for the existence and global stability of a unique equilibrium of this class of fuzzy BAM neural networks with Lipschitzian activation functions without assuming their boundedness, monotonicity or differentiability and subjected to impulsive state displacements at fixed instants of time. Some numerical examples are given to demonstrate the effectiveness of the obtained results.

Keywords—Discrete-time fuzzy BAM neural networks; Impulses; Global exponential stability; Global asymptotical stability; Equilibrium point.

I. Introduction

ARTIFICIAL neural networks, a new method for various information processing, is now widely used in the fields of pattern recognition, image processing, association, optimal computation, and others. Such applications heavily depend on the dynamical behaviors of the networks such as stability, convergence, oscillatory properties, and so on. Neural networks and their various generalizations have attracted the attention of the scientific community due to their promising potential for tasks of classification, associative memory, and parallel computation and their ability to solve difficult optimization problems. Such applications rely on the existence of equilibrium points and qualitative properties of the neural networks. Recently, a class of two-layer heteroassociative networks called bidirectional associative memory (BAM) networks [1-5] with or without axonal signal transmission delays has been proposed and used in many fields such as pattern recognition and automatic control. Kosko [1-3], Li [6], [7], Cao [8], studied the stability of BAM neural network with or without delays.

However, besides delay effect, impulsive effect likewise exists in neural networks [9]. For instance, in implementation of electronic networks, the state of the networks is subject to instantaneous perturbations and experiences abrupt change at certain instants, which may be caused by switching phenomenon, frequency change or other sudden noise, that is, does exhibit impulsive effects. Therefore, it is necessary to consider both impulsive effect and delay effect on neural networks, see [9-16] and the references therein. Though the BAM non-impulsive systems have been well studied in theory and in practice (for example see [17-19] and references cited therein), the theory of impulsive differential equations is not only being recognized to be richer than the corresponding theory of differential equations, but also represents a more natural framework for mathematical modeling of many real-world phenomena, such as population dynamic and neural networks. In [20], Li provided some sufficient conditions for the existence and the global exponential stability of a BAM networks with Lipschitzian activation functions without assuming their boundedness, monotonicity and subjected to impulsive state displacements at fixed instants of time. The global asymptotic stability of delay bidirectional associative memory neural networks with impulses are established by constructing suitable Lyapunov functional in [21]. More recently, Lou and Cui [22] studied the global asymptotic stability of delay BAM neural networks with impulses.

On the other hand, in mathematical modeling of real world problems, we encounter some inconveniences besides impulses and delays, namely, the complexity and the uncertainty or vagueness. Vagueness is opposite to exactness and we argue that it cannot be avoided in the human way of regarding the world. Any attempt to explain an extensive detailed description necessarily leads to using vague concepts since precise description contains abundant number of details. To understand it, we must group them together-and this can hardly be done precisely. A non-substitutable role is here played by natural language. For the sake of taking vagueness into consideration, fuzzy theory is viewed as a more suitable setting. Based on traditional CNNs, Yang and Yang [23] first introduced the fuzzy cellular neural networks (FCNNs), which integrates fuzzy logic into the structure of traditional CNNs and maintains local connectedness among cells. Unlike previous CNNs, FCNN is a very useful paradigm for image processing problems, which has fuzzy logic between its template input and/or output besides the sum of product operation. Studies have shown that the FCNN is very useful paradigm for image processing problems and some results on stability have been derived for the FCNN, see [24-27].

As pointed out in [28-31], some implementations of the continuous-time neural networks, it is essential to formulate a BAM fuzzy discrete-time system that is an analogue of the continuous-time system without fuzzy logic. Therefore, it is both theoretical and practical importance to study the dynamics of fuzzy BAM discrete-time neural networks. To the

Chao Wang is with the Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, People's Republic of China.

Yongkun Li is with the Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, People's Republic of China.
E-mail: yklie@ynu.edu.cn.

best of our knowledge, few authors have considered discrete-time fuzzy cellular neural networks (DTFCNN) with variable delays and impulses [32]. Motivated by the above discussion, in this paper, by applying the similar mathematical analysis techniques in [20], we investigate the following discrete-time BAM fuzzy neural networks with impulses:

$$\begin{cases} \Delta x_i(n) = -a_i x_i(n) + \sum_{j=1}^{s} b_{ij}u_j + \sum_{j=1}^{s} a_{ij} f_j(y_j(n)) \\ \qquad + \bigwedge_{j=1}^{s} c_{ij} f_j(y_j(n-\tau_{ij})) \\ \qquad + \bigvee_{j=1}^{s} e_{ij} f_j(y_j(n-\tau_{ij})) + \bigwedge_{j=1}^{s} T_{ij}u_j \\ \qquad + \bigvee_{j=1}^{s} H_{ij}u_j + c_i,\ n \neq n_k,\ i = 1,2,\ldots,r, \\ \Delta x_i(n_k) = I_k(x_i(n_k)),\ n = n_k, \\ \qquad i = 1,2,\ldots,r,\ k \in \mathbb{Z}^+, \\ \Delta y_j(n) = -b_j y_j(n) + \sum_{i=1}^{r} m_{ji}v_i + \sum_{i=1}^{r} d_{ji} g_i(x_i(n)) \\ \qquad + \bigwedge_{i=1}^{r} \alpha_{ji} g_i(x_i(n-\sigma_{ji})) \\ \qquad + \bigvee_{i=1}^{r} \beta_{ji} g_i(x_i(n-\sigma_{ji})) + \bigwedge_{i=1}^{r} T_{ji}v_i \\ \qquad + \bigvee_{i=1}^{r} H_{ji}v_i + d_j,\ n \neq n_k,\ j = 1,2,\ldots,s, \\ \Delta y_j(n_k) = J_k(y_j(n_k)),\ n = n_k, \\ \qquad j = 1,2,\ldots,s,\ k \in \mathbb{Z}^+. \end{cases} \quad (1)$$

For integers a and b with $a < b, N[a,b]$ denotes the discrete interval given by $N[a,b] = \{a, a+1, \ldots, b-1, b\}$. $C(N[a,b],\mathbb{R})$ denotes the set of all functions $\varphi : N[a,b] \to \mathbb{R}$. In system (1), $x_i(n), y_j(n)$ are the states of the ith neuron from the neural field F_X and the jth neuron from the neural field F_Y at time m, respectively; f_j, g_i denote the activation functions of the ith neuron from the neural field F_X and the jth neuron from the neural field F_Y, respectively; τ_{ij}, σ_{ji} are the transmission delays which are non-negative constants satisfying $\tau_{ij} \in N[0,+\infty)$, $\sigma_{ji} \in N[0,+\infty)$, b_{ij}, m_{ji} are elements of feed-forward template, a_{ij}, d_{ji} are elements of feed-back template, c_{ij}, α_{ji} are elements of fuzzy feedback MIN template, e_{ij}, β_{ji} are elements of fuzzy feedback MAX template, T_{ij} and H_{ij} are elements of fuzzy feed-forward MIN template and fuzzy feed-forward MAX template, respectively; $\bigvee$ and $\bigwedge$ denote the fuzzy AND and fuzzy OR operation, respectively; c_i, v_i and d_j, u_j denote external inputs and biases of the ith neuron from the neural field F_X and the jth neuron from the neural field F_Y, respectively; $\Delta x_i(n) = x_i(n+1) - x_i(n)$, $\Delta y_j(n) = y_j(n+1) - y_j(n)$, $a_i, b_j \in (0,1)$; $\Delta x_i(n_k) = x_i(n_k+1) - x_i(n_k), \Delta y_j(n_k) = y_j(n_k+1) - y_j(n_k)$, $\{n_l\}$ is a sequence real numbers such that $0 < n_1 < n_2 < \cdots < n_l \to \infty$ as $l \to \infty$.

The system (1) is supplemented with the initial values given by

$$\begin{cases} x_i(\xi) = \varphi_i(\xi), \xi \in N[-\sigma, 0], \sigma = \max_{1\le i\le r, 1\le j\le s}\{\sigma_{ji}\}, \\ y_j(\xi) = \psi_j(\xi), \xi \in N[-\tau, 0], \tau = \max_{1\le i\le r, 1\le j\le s}\{\tau_{ij}\}, \end{cases}$$

where $\varphi_i \in C(N[-\sigma,0],\mathbb{R}), \psi_j \in C(N[-\tau,0],\mathbb{R})$.

Throughout this paper, we assume that:

(H_1) a_i, $b_j \in (0,1)$, a_{ij}, d_{ji}, c_{ij}, e_{ij}, α_{ji}, β_{ji}, c_i, d_j are constants, τ_{ij}, $\sigma_{ji} \in N[0,+\infty)$, $i = 1,2,\ldots,r$, $j = 1,2,\ldots,s$.

(H_2) f_j, $g_i \in C(\mathbb{R},\mathbb{R})$ and there exists positive number L_j^f, L_i^g such that

$$|f_j(x) - f_j(y)| \le L_j^f |x-y|$$

and

$$|g_i(x) - g_i(y)| \le L_i^g |x-y|$$

for all $i = 1,2,\ldots,r,\ j = 1,2,\ldots,s$.

II. Preliminaries

In this section, we shall first recall some definitions, basic lemmas which are useful in our proofs.

Definition 1. *The equilibrium point* $z^* = (x_1^*, \ldots, x_r^*, y_1^*, \ldots, y_s^*)^T$ *of system* (1) *is said to be globally exponentially stable, if there exist constants* $\lambda > 0$ *and* $\delta > 0$ *such that*

$$\|z(n) - z^*\| \le \frac{\delta}{\lambda^n}(\|\varphi - x^*\| + \|\psi - y^*\|),$$

where $z(n) = (x_1(n), \ldots, x_r(n), y_1(n), \ldots, y_s(n))^T$ *is any solution of system* (1) *with initial value* $\phi(\xi) = (\varphi_1(\xi), \ldots, \varphi_r(\xi), \psi_1(\xi), \ldots, \psi_s(\xi))^T$ *and*

$$\|\varphi - x^*\| = \sup_{\xi \in N[-\sigma,0]} \left\{ \sum_{i=1}^{r} |\varphi(\xi) - x_i^*| \right\},$$

$$\|\psi - y^*\| = \sup_{\xi \in N[-\tau,0]} \left\{ \sum_{j=1}^{s} |\psi(\xi) - y_i^*| \right\}.$$

Lemma 1. *([23]). Let* $x = (x_1, x_2, \ldots, x_n)^T$ *and* $x' = (x_1', x_2', \ldots, x_n')^T$ *be two states of system* (1)*, then we have*

$$\left| \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j) - \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j') \right| \le \sum_{j=1}^{n} |\alpha_{ij}| |f_j(x_j) - f_j(x_j')|$$

and

$$\left| \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j) - \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j') \right| \le \sum_{j=1}^{n} |\beta_{ij}| |f_j(x_j) - f_j(x_j')|.$$

III. Existence and uniqueness of equilibria

In this section, we will derive sufficient conditions for the existence of equilibria on the coefficients and the activation functions in (1). An equilibrium solution of (1) is a constant vector $(x_1^*, x_2^*, \ldots, x_r^*, y_1^*, y_2^*, \ldots, y_s^*) \in \mathbb{R}^{r+s}$ which satisfies the system

$$\begin{cases} a_i x_i^* = \sum_{j=1}^{s} a_{ij} f_j(y_j^*) + \bigwedge_{j=1}^{s} c_{ij} f_j(y_j^*) \\ \qquad + \bigvee_{j=1}^{s} e_{ij} f_j(y_j^*) + \varrho_i, i = 1, 2, \ldots, r, \\ b_j y_j^* = \sum_{i=1}^{r} d_{ji} g_i(x_i^*) + \bigwedge_{i=1}^{r} \alpha_{ji} g_i(x_i^*) \\ \qquad + \bigvee_{i=1}^{r} \beta_{ji} g_i(x_i^*) + \theta_j, j = 1, 2, \ldots, s, \end{cases} \tag{2}$$

where $\varrho_i = \sum_{j=1}^{s} b_{ij} u_j + \bigwedge_{j=1}^{s} T_{ij} u_j + \bigvee_{j=1}^{s} H_{ij} u_j + c_i$, $\theta_j = \sum_{i=1}^{r} m_{ji} v_i + \bigwedge_{i=1}^{r} T_{ji} v_i + \bigvee_{i=1}^{r} H_{ji} v_i + d_j$, $j = 1, 2, \ldots, s$, $i = 1, 2, \ldots, r$, when the impulsive jumps $I_k(\cdot), \ J_k(\cdot)$ as assumed to satisfy $I_k(x_i^*) = 0, \ J_k(y_j^*) = 0, \ i = 1, 2, \ldots, r, \ j = 1, 2, \ldots, s, \ k \in \mathbb{Z}^+$, where $\mathbb{Z}^+$ denotes the set of all positive integers. In what follows, we use the following norm of $\mathbb{R}^{r+s}$:

$$\|z\| = \sum_{l=1}^{r+s} |z_l|, \quad \text{for} \quad z = (z_1, z_2, \ldots, z_{r+s})^T \in \mathbb{R}^{r+s}.$$

Theorem 1. *Suppose that* (H_3)

$$a_i > L_i^g \sum_{j=1}^{s} (|d_{ji}| + |\alpha_{ji}| + |\beta_{ji}|), \ i = 1, 2, \ldots, r,$$

$$b_j > L_j^f \sum_{i=1}^{r} (|a_{ij}| + |c_{ij}| + |e_{ij}|), \ j = 1, 2, \ldots, s.$$

Then there exists a unique solution of the system (2).

Proof: It follows from (H_3) that

$$\max_{1 \le j \le s} \left(\frac{L_j^f}{b_j} \sum_{i=1}^{r} (|a_{ij}| + |c_{ij}| + |e_{ij}|) \right) < 1,$$

and

$$\max_{1 \le i \le r} \left(\frac{L_i^g}{a_i} \sum_{j=1}^{s} (|d_{ji}| + |\alpha_{ji}| + |\beta_{ji}|) \right) < 1.$$

Define θ as

$$\theta = \max\Big\{ \max_{1 \le j \le s} \left(\frac{L_j^f}{b_j} \sum_{i=1}^{r} (|a_{ij}| + |c_{ij}| + |e_{ij}|) \right), \max_{1 \le i \le r} \left(\frac{L_i^g}{a_i} \sum_{j=1}^{s} (|d_{ji}| + |\alpha_{ji}| + |\beta_{ji}|) \right) \Big\}.$$

Let $a_i x_i^* = u_i^*, \ b_j y_j^* = v_j^*$, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, s$ in (2), then we get

$$\begin{cases} u_i^* = \sum_{j=1}^{s} a_{ij} f_j\left(\frac{v_j^*}{b_j}\right) + \bigwedge_{j=1}^{s} c_{ij} f_j\left(\frac{v_j^*}{b_j}\right) \\ \qquad + \bigvee_{j=1}^{s} e_{ij} f_j\left(\frac{v_j^*}{b_j}\right) + \varrho_i, i = 1, 2, \ldots, r, \\ v_j^* = \sum_{i=1}^{r} d_{ji} g_i\left(\frac{u_i^*}{a_i}\right) + \bigwedge_{i=1}^{r} \alpha_{ji} g_i\left(\frac{u_i^*}{a_i}\right) \\ \qquad + \bigvee_{i=1}^{r} \beta_{ji} g_i\left(\frac{u_i^*}{a_i}\right) + \theta_j, j = 1, 2, \ldots, s. \end{cases} \tag{3}$$

To finish the proof, it suffices to show that (3) has a unique solution. Consider a mapping $\Phi : \mathbb{R}^{r+s} \to \mathbb{R}^{r+s}$ defined by

$$\Phi(u_1, \ldots, u_r, v_1, \ldots, v_s) = \begin{pmatrix} \sum_{j=1}^{s} a_{1j} f_j\left(\frac{v_j}{b_j}\right) + \bigwedge_{j=1}^{s} c_{1j} f_j\left(\frac{v_j}{b_j}\right) \\ \vdots \\ \sum_{j=1}^{s} a_{rj} f_j\left(\frac{v_j}{b_j}\right) + \bigwedge_{j=1}^{s} c_{rj} f_j\left(\frac{v_j}{b_j}\right) \\ \sum_{i=1}^{r} d_{1i} g_i\left(\frac{u_i}{a_i}\right) + \bigwedge_{i=1}^{r} \alpha_{1i} g_i\left(\frac{u_i}{a_i}\right) \\ \vdots \\ \sum_{i=1}^{r} d_{si} g_i\left(\frac{u_i}{a_i}\right) + \bigwedge_{i=1}^{r} \alpha_{si} g_i\left(\frac{u_i}{a_i}\right) \end{pmatrix} + \begin{pmatrix} \bigvee_{j=1}^{s} e_{1j} f_j\left(\frac{v_j}{b_j}\right) + \varrho_1 \\ \bigvee_{j=1}^{s} e_{rj} f_j\left(\frac{v_j}{b_j}\right) + \varrho_r \\ \bigvee_{i=1}^{r} \beta_{1i} g_i\left(\frac{u_i}{a_i}\right) + \theta_1 \\ \bigvee_{i=1}^{r} \beta_{si} g_i\left(\frac{u_i}{a_i}\right) + \theta_s \end{pmatrix}.$$

We show that $\Phi : \mathbb{R}^{r+s} \to \mathbb{R}^{r+s}$ is a global contraction on $\mathbb{R}^{r+s}$ endowed with the norm $\| \cdot \|$. In fact, for $\overline{u} = (\overline{u_1}, \ldots, \overline{u_r}, \overline{v_1}, \ldots, \overline{v_s})^T$, $u = (u_1, \ldots, u_r, v_1, \ldots, v_s)^T \in \mathbb{R}^{r+s}$, one has

$$\begin{aligned} &\|\Phi(\overline{u}) - \Phi(u)\| \\ &= \|\Phi(\overline{u_1}, \ldots, \overline{u_r}, \overline{v_1}, \ldots, \overline{v_s}) - \Phi(u_1, \ldots, u_r, v_1, \ldots, v_s)\| \\ &= \sum_{i=1}^{r} \Bigg| \sum_{j=1}^{s} a_{ij} \left[f_j\left(\frac{\overline{v_j}}{b_j}\right) - f_j\left(\frac{v_j}{b_j}\right) \right] \\ &\quad + \left[\bigwedge_{j=1}^{s} c_{ij} f_j\left(\frac{\overline{v_j}}{b_j}\right) - \bigwedge_{j=1}^{s} c_{ij} f_j\left(\frac{v_j}{b_j}\right) \right] \\ &\quad + \left[\bigvee_{j=1}^{s} e_{ij} f_j\left(\frac{\overline{v_j}}{b_j}\right) - \bigvee_{j=1}^{s} e_{ij} f_j\left(\frac{v_j}{b_j}\right) \right] \Bigg| \\ &\quad + \sum_{j=1}^{s} \Bigg| \sum_{i=1}^{r} d_{ji} \left[g_i\left(\frac{\overline{u_i}}{a_i}\right) - g_i\left(\frac{u_i}{a_i}\right) \right] \\ &\quad + \left[\bigwedge_{i=1}^{r} \alpha_{ji} g_i\left(\frac{\overline{u_i}}{a_i}\right) - \bigwedge_{i=1}^{r} \alpha_{ji} g_i\left(\frac{u_i}{a_i}\right) \right] \\ &\quad + \left[\bigvee_{i=1}^{r} \beta_{ji} g_i\left(\frac{\overline{u_i}}{a_i}\right) - \bigvee_{i=1}^{r} \beta_{ji} g_i\left(\frac{u_i}{a_i}\right) \right] \Bigg| \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{r}\Bigg[\sum_{j=1}^{s}|a_{ij}|L_j^f\left|\frac{\overline{v_j}-v_j}{b_j}\right| \\
&\quad+\sum_{j=1}^{s}|c_{ij}|\left|f_j\left(\frac{\overline{v_j}}{b_j}\right)-f_j\left(\frac{v_j}{b_j}\right)\right| \\
&\quad+\sum_{j=1}^{s}|e_{ij}|\left|f_j\left(\frac{\overline{v_j}}{b_j}\right)-f_j\left(\frac{v_j}{b_j}\right)\right|\Bigg] \\
&\quad+\sum_{j=1}^{s}\Bigg[\sum_{i=1}^{r}|d_{ji}|L_i^g\left|\frac{\overline{u_i}-u_i}{a_i}\right| \\
&\quad+\sum_{i=1}^{r}|\alpha_{ji}|\left|g_i\left(\frac{\overline{u_i}}{a_i}\right)-g_i\left(\frac{u_i}{a_i}\right)\right| \\
&\quad+\sum_{i=1}^{r}|\beta_{ji}|\left|g_i\left(\frac{\overline{u_i}}{a_i}\right)-g_i\left(\frac{u_i}{a_i}\right)\right|\Bigg] \\
&\le \sum_{i=1}^{r}\sum_{j=1}^{s}(|a_{ij}|+|c_{ij}|+|e_{ij}|)L_j^f\left|\frac{\overline{v_j}-v_j}{b_j}\right| \\
&\quad+\sum_{j=1}^{s}\sum_{i=1}^{r}(|d_{ji}|+|\alpha_{ji}|+|\beta_{ji}|)L_i^g\left|\frac{\overline{u_i}-u_i}{a_i}\right| \\
&\le \left[\max_{1\le j\le s}\left(\frac{L_j^f}{b_j}\sum_{i=1}^{r}(|a_{ij}|+|c_{ij}|+|e_{ij}|)\right)\right]\sum_{j=1}^{s}|\overline{v_j}-v_j| \\
&\quad+\left[\max_{1\le i\le r}\left(\frac{L_i^g}{a_i}\sum_{j=1}^{s}(|d_{ji}|+|\alpha_{ji}|+|\beta_{ji}|)\right)\right]\sum_{i=1}^{r}|\overline{u_i}-u_i| \\
&\le \theta\left(\sum_{j=1}^{s}|\overline{v_j}-v_j|+\sum_{i=1}^{r}|\overline{u_i}-u_i|\right) \\
&\le \theta\|\overline{u}-u\|.
\end{aligned}
$$

By our hypothesis $\theta < 1$ which implies that $\Phi : \mathbb{R}^{r+s} \to \mathbb{R}^{r+s}$ is a contraction on $\mathbb{R}^{r+s}$. Hence by the contraction mapping principle, there exists a unique fixed point of the map $\Phi : \mathbb{R}^{r+s} \to \mathbb{R}^{r+s}$ which is a solution of system (3) from which the existence of a unique solution of (2) will follow. The proof is complete. ■

IV. Stability of equilibria of fuzzy BAM neural networks with impulses

In this section, we derive sufficient conditions for the stability of equilibria of (1).

Theorem 2. *Assume that $(H_1)-(H_3)$ hold. Futhermore, suppose that the impulsive operator $I_k(x_i(n))$ and $J_k(y_j(n))$ satisfy*

$$
\begin{cases}
I_k(x_i(n_k)) = -\gamma_{ik}(x_i(n_k)-x^*), i=1,2,\ldots,r,\ k\in\mathbb{Z}^+,\\
J_j(y_j(n_k)) = -\overline{\gamma_{jk}}(y_j(n_k)-y_j^*), j=1,2,\ldots,s,\ k\in\mathbb{Z}^+,
\end{cases}
\tag{4}
$$

where $(x_1^*, x_2^*, \ldots, x_r^*, y_1^*, y_2^*, \ldots, y_s^*)^T \in \mathbb{R}^{r+s}$ is the unique solution of the system (2)*, and*

(H_4) *for* $k \in \mathbb{Z}^+$,

$$\sum_{i=1}^{r}\left[|1-\gamma_{ik}|+\sum_{j=1}^{s}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g\right]\le r,$$

$$\sum_{j=1}^{s}\left[|1-\overline{\gamma_{jk}}|+\sum_{i=1}^{r}(|c_{ij}|+|e_{ij}|)L_j^f\right]\le s.$$

Then it is also a equilibrium solution of system (1) *and it is globally asymptotically stable.*

Proof: According to Theorem 1, system (2) has a unique equilibrium $(x_1^*, x_2^*, \ldots, x_r^*, y_1^*, y_2^*, \ldots, y_s^*)^T \in \mathbb{R}^{r+s}$. By (4), we have $I_k(x_i^*) = 0$, $J_k(v_j^*) = 0$, $i = 1,2,\ldots,r$, $j = 1,2,\ldots,s$, $k \in \mathbb{Z}^+$. Hence $(x_1^*, \ldots, x_r^*, y_1^*, \ldots, y_s^*)^T \in \mathbb{R}^{r+s}$ is also the unique equilibrium point of (1). Let $(x_1(n), \ldots, x_r(n), y_1(n), \ldots, y_s(n))^T$ be an arbitrary solution of (1) and take $\mu_i(n) = x_i(n) - x_i^*$, $n > -\sigma$, $\eta_j(n) = y_j(n) - y_j^*$, $n > -\tau$, then it follows from (1) that

$$
\begin{cases}
\Delta\mu_i(n) =- a_i\mu_i(n)+\sum_{j=1}^{s}a_{ij}(f_j(y_j(n))-f_j(y_j^*))\\
\qquad+\left(\bigwedge_{j=1}^{s}c_{ij}f_j(y_j(n-\tau_{ij}))-\bigwedge_{j=1}^{s}c_{ij}f_j(y_j^*)\right)\\
\qquad+\left(\bigvee_{j=1}^{s}e_{ij}f_j(y_j(n-\tau_{ij}))-\bigvee_{j=1}^{s}e_{ij}f_j(y_j^*)\right),\\
\qquad n\ne n_k,\ i=1,2,\ldots,r,\\
\Delta\mu_i(n_k) =I_k(\mu_i(n_k))=-\gamma_{ik}\mu_i(n_k),\\
\qquad n=n_k,\ i=1,2,\ldots,r,\ k\in\mathbb{Z}^+,\\
\Delta\eta_j(n) =- b_j\eta_j(n)+\sum_{i=1}^{r}d_{ji}(g_i(x_i(n))-g_i(x_i^*))\\
\qquad+\left(\bigwedge_{i=1}^{r}\alpha_{ji}g_i(x_i(n-\sigma_{ji}))-\bigwedge_{i=1}^{r}\alpha_{ji}g_i(x_i^*)\right)\\
\qquad+\left(\bigvee_{i=1}^{r}\beta_{ji}g_i(x_i(n-\sigma_{ji}))-\bigvee_{i=1}^{r}\beta_{ji}g_i(x_i^*)\right),\\
\qquad n\ne n_k,\ j=1,2,\ldots,s,\\
\Delta\eta_j(n_k) =J_k(\eta_j(n_k))=-\overline{\gamma_{jk}}\eta_j(n_k),\\
\qquad n=n_k,\ j=1,2,\ldots,s,\ k\in\mathbb{Z}^+.
\end{cases}
\tag{5}
$$

By $0 < a_i,\ b_j < 1$, $i = 1,2,\ldots,r$, $j = 1,2,\ldots,s$, we obtain

$$
\begin{aligned}
|\mu_i(n+1)| &\le (1-a_i)|\mu_i(n)|+\sum_{j=1}^{s}|a_{ij}||f_j(y_j(n))-f_j(y_j^*)| \\
&\quad+\left|\bigwedge_{j=1}^{s}c_{ij}f_j(y_j(n-\tau_{ij}))-\bigwedge_{j=1}^{s}c_{ij}f_j(y_j^*)\right| \\
&\quad+\left|\bigvee_{j=1}^{s}e_{ij}f_j(y_j(n-\tau_{ij}))-\bigvee_{j=1}^{s}e_{ij}f_j(y_j^*)\right| \\
&\le (1-a_i)|\mu_i(n)|+\sum_{j=1}^{s}|a_{ij}|L_j^f|y_j(n)-y_j^*| \\
&\quad+\sum_{j=1}^{s}|c_{ij}|L_j^f|y_j(n-\tau_{ij})-y_j^*| \\
&\quad+\sum_{j=1}^{s}|e_{ij}|L_j^f|y_j(n-\tau_{ij})-y_j^*|
\end{aligned}
$$

$$= (1-a_i)|\mu_i(n)| + \sum_{j=1}^{s}|a_{ij}|L_j^f|\eta_j(n)| + \sum_{j=1}^{s}(|c_{ij}|+|e_{ij}|)L_j^f|\eta_j(n-\tau_{ij})|.$$

Thus,

$$\Delta|\mu_i(n)| \le -a_i|\mu_i(n)| + \sum_{j=1}^{s}|a_{ij}|L_j^f|\eta_j(n)| + \sum_{j=1}^{s}(|c_{ij}|+|e_{ij}|)L_j^f|\eta_j(n-\tau_{ij})|,$$
$$n>0,\ n\neq n_k,\ i=1,2,\dots,r.$$

And similarly, we have

$$\Delta|\eta_j(n)| \le -b_j|\eta_j(n)| + \sum_{i=1}^{r}|d_{ji}|L_i^g|\mu_i(n)| + \sum_{i=1}^{r}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g|\mu_i(n-\sigma_{ji})|,$$
$$n>0,\ n\neq n_k,\ j=1,2,\dots,s.$$

Define Lyapunov function $V(n)$ as follows:

$$V(n) = \sum_{i=1}^{r}\Big[|\mu_i(n)| + \sum_{j=1}^{s}(|c_{ij}|+|e_{ij}|) \times L_j^f\Big(\sum_{l=n-\tau_{ij}}^{n-1}|\eta_j(l)|\Big)\Big] + \sum_{j=1}^{s}\Big[|\eta_j(n)| + \sum_{i=1}^{r}(|\alpha_{ji}|+|\beta_{ji}|) \times L_i^g\Big(\sum_{l=n-\sigma_{ji}}^{n-1}|\mu_i(l)|\Big)\Big].$$

It is easy to check that $V(n)>0$ for $n>0$ and $V(0)$ is positive and finite. Calculating the defference of V along solutions of system (5), we obtain

$$\begin{aligned}
&\Delta V(n)\\
&= \sum_{i=1}^{r}\Big[\Delta|\mu_i(n)| + \sum_{j=1}^{s}(|c_{ij}|+|e_{ij}|)L_j^f(|\eta_j(n)| - |\eta_j(n-\tau_{ij})|)\Big] + \sum_{j=1}^{s}\Big[\Delta|\eta_j(n)| + \sum_{i=1}^{r}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g(|\mu_i(n)|-|\mu_i(n-\sigma_{ji})|)\Big]\\
&\le \sum_{i=1}^{r}\Big[-a_i|\mu_i(n)| + \sum_{j=1}^{s}(|a_{ij}|+|c_{ij}|+|e_{ij}|)L_j^f|\eta_j(n)|\Big] + \sum_{j=1}^{s}\Big[-b_j|\eta_j(n)| + \sum_{i=1}^{r}(|d_{ji}|+|\alpha_{ji}|+|\beta_{ji}|)L_i^g|\mu_i(n)|\Big]\\
&= \sum_{i=1}^{r}\Big[-a_i + \sum_{j=1}^{s}(|d_{ji}|+|\alpha_{ji}|+|\beta_{ji}|)L_i^g\Big]|\mu_i(n)| + \sum_{j=1}^{s}\Big[-b_j + \sum_{i=1}^{r}(|a_{ij}|+|c_{ij}|+|e_{ij}|)L_j^f\Big]|\eta_j(n)|,
\end{aligned}$$

by (H_3), we have

$$a_i > L_i^g\sum_{j=1}^{s}(|d_{ji}|+|\alpha_{ji}|+|\beta_{ji}|),\ i=1,2,\dots,r,$$

$$b_j > L_j^f\sum_{i=1}^{r}(|a_{ij}|+|c_{ij}|+|e_{ij}|),\ j=1,2,\dots,s,$$

it follows that

$$\Delta V(n)\le 0,\ \text{for } n>0,\ n\neq n_k.$$

Also, from (5) and (H_4), we get

$$\begin{cases}\Delta|\mu_i(n_k)| = (|1-\gamma_{ik}|-1)|\mu_i(n_k)|, \\ \qquad i=1,2,\dots,r,\ k\in\mathbb{Z}^+,\\ \Delta|\eta_j(n_k)| = (|1-\overline{\gamma_{jk}}|-1)|\eta_j(n_k)|,\\ \qquad j=1,2,\dots,s,\ k\in\mathbb{Z}^+\end{cases}$$

and

$$\begin{aligned}
&\Delta V(n_k)\\
&= \sum_{i=1}^{r}\Big[\Delta|\mu_i(n_k)| + \sum_{j=1}^{s}(|c_{ij}|+|e_{ij}|)L_j^f(|\eta_j(n_k)| - |\eta_j(n_k-\tau_{ij})|)\Big] + \sum_{j=1}^{s}\Big[\Delta|\eta_j(n_k)| + \sum_{i=1}^{r}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g(|\mu_i(n_k)|-|\mu_i(n_k-\sigma_{ji})|)\Big]\\
&\le \sum_{i=1}^{r}\Big[(|1-\gamma_{ik}|-1)|\mu_i(n_k)| + \sum_{j=1}^{s}(|c_{ij}|+|e_{ij}|)L_j^f|\eta_j(n_k)|\Big] + \sum_{j=1}^{s}\Big[(|1-\overline{\gamma_{jk}}|-1)|\eta_j(n_k)| + \sum_{i=1}^{r}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g|\mu_i(n_k)|\Big]\\
&= \sum_{i=1}^{r}\Big[(|1-\gamma_{ik}|-1) + \sum_{j=1}^{s}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g\Big]|\mu_i(n_k)| + \sum_{j=1}^{s}\Big[(|1-\overline{\gamma_{jk}}|-1) + \sum_{i=1}^{r}(|c_{ij}|+|e_{ij}|)L_j^f\Big]|\eta_j(n_k)|\\
&\le 0.
\end{aligned}$$

Hence, we have

$$\Delta V(n)\le 0,\ \text{for } n>0.$$

It follows that the origin of (5) is global asympototically stable, which implies that equilibrium solution $(x_1^*,\dots,x_r^*,y_1^*,\dots,y_s^*)\in\mathbb{R}^{r+s}$ of system (1) is globally asymptotically stable. The proof is complete. ■

Theorem 3. *Assume that* $(H_1)-(H_3)$ *hold. Furthermore, suppose that the impulsive operators* $I_k(x_i(n))$ *and* $J_k(y_j(n)$ *satisfy*

$$\begin{cases} I_k(x_i(n_k)) \;=\; -\gamma_{ik}(x_i(n_k)-x_i^*),\, a_i \le \gamma_{ik} \le 2-a_i, \\ \qquad\qquad\qquad i=1,2,\dots,r,\, k\in\mathbb{Z}^+, \\ J_j(y_j(n_k)) \;=\; -\overline{\gamma_{jk}}(y_j(n_k)-y_j^*),\, b_j \le \overline{\gamma_{jk}} \le 2-b_j, \\ \qquad\qquad\qquad j=1,2,\dots,s,\, k\in\mathbb{Z}^+. \end{cases} \tag{6}$$

Then system (1) *has a unique equilibrium point* $(x_1^*,\dots,x_r^*, y_1^*,\dots,y_s^*)^T \in \mathbb{R}^{r+s}$*, which is globally exponentially stable in the sense that there exit constant* $\lambda>0$ *and* $\delta>0$ *such that*

$$\sum_{i=1}^{r}|x_i(n)-x_i^*|+\sum_{j=1}^{s}|y_j(n)-y_j^*|$$
$$\le \frac{\delta}{\lambda^n}\Bigg[\sum_{i=1}^{r}\sup_{s\in[-\sigma,0]}|x_i(s)-x_i^*|+\sum_{j=1}^{s}\sup_{s\in[-\tau,0]}|y_j(s)-y_j^*|\Bigg].$$

Proof: According to Theorem 1, system (2) has a unique solution $(x_1^*,\dots,x_r^*,y_1^*,\dots,y_s^*)^T\in\mathbb{R}^{r+s}$. By (6), we have $I_k(x_i^*)=0,\ J_k(y_j^*)=0,\ i=1,2,\dots,r,\ j=1,2,\dots,s,\ k\in\mathbb{Z}^+$. Hence, $(x_1^*,\dots,x_r^*,y_1^*,\dots,y_s^*)^T\in\mathbb{R}^{r+s}$ is also the unique equilibrium point of system (1). Let $(x_1(n),\dots,x_r(n),y_1(n),\dots,y_s(n))^T$ be an arbitrary solution of (1), then it follows from (1) and $0\le a_i,\, b_j\le 1,\ i=1,2,\dots,r,\ j=1,2,\dots,s$ that

$$\begin{aligned} &|x_i(n+1)-x_i^*| \\ \le\ & (1-a_i)|x_i(n)-x_i^*|+\sum_{j=1}^{s}|a_{ij}|L_j^f|y_j(n)-y_j^*| \\ &+\sum_{j=1}^{s}(|c_{ij}|+|e_{ij}|)L_j^f|y_j(n-\tau_{ij})-y_j^*|, \\ &n>0, n\ne n_k, i=1,2,\dots,r, \end{aligned} \tag{7}$$

$$\begin{aligned} &|y_j(n+1)-y_j^*| \\ \le\ & (1-b_j)|y_j(n)-y_j^*|+\sum_{i=1}^{r}|d_{ji}|L_i^g|x_i(n)-x_i^*| \\ &+\sum_{i=1}^{r}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g|x_i(n-\sigma_{ji})-x_i^*|, \\ &n>0, n\ne n_k, j=1,2,\dots,s. \end{aligned} \tag{8}$$

From (H_3), there exist constants $\rho_i,\, \xi_j>0$ such that

$$a_i\rho_i - L_i^g\sum_{j=1}^{s}\xi_j(|d_{ji}|+|\alpha_{ji}|+|\beta_{ji}|)>0,\ i=1,2,\dots,r,$$

$$b_j\xi_j - L_j^f\sum_{i=1}^{r}\rho_i(|a_{ij}|+|c_{ij}|+|e_{ij}|)>0,\ j=1,2,\dots,s.$$

Now, we consider the functions $\mathscr{A}_i(\cdot),\ \mathscr{B}_j(\cdot)$ defined by

$$\mathscr{A}_i(\theta_i)=\rho_i-\rho_i(1-a_i)\theta_i-L_i^g\sum_{j=1}^{s}\xi_j(|d_{ji}|+|\alpha_{ji}|+|\beta_{ji}|)\theta_i^{1+\sigma_{ji}},$$

$$\mathscr{B}_j(\zeta_j)=\xi_j-\xi_j(1-b_j)\zeta_j-L_j^f\sum_{i=1}^{r}\rho_i(|a_{ij}|+|c_{ij}|+|e_{ij}|)\zeta_j^{1+\tau_{ij}},$$

respectively, where $\theta_i,\,\zeta_j\in[1,+\infty), i=1,2,\dots,r, j=1,2,\dots,s$. It is clear that

$$\mathscr{A}_i(1)=a_i\rho_i-L_i^g\sum_{j=1}^{s}\xi_j(|d_{ji}|+|\alpha_{ji}|+|\beta_{ji}|)>0,$$

$$\mathscr{B}_j(1)=b_j\xi_j-L_j^f\sum_{i=1}^{r}\rho_i(|a_{ij}|+|c_{ij}|+|e_{ij}|)>0.$$

Since $\mathscr{A}_i(\cdot),\ \mathscr{B}_j(\cdot)$ are continuous on $[1,+\infty)$ and $\mathscr{A}_i(\theta_i)\to-\infty,\ \mathscr{B}_j(\zeta_j)\to-\infty$ as $\theta_i\to+\infty,\ \zeta_j\to+\infty$, there exist $\theta_i^*,\ \zeta_j^*\in(1,+\infty)$ such that $\mathscr{A}_i(\theta_i^*)=0,\ \mathscr{B}_j(\zeta_j^*)=0$ and $\mathscr{A}_i(\theta_i)>0,\ \mathscr{B}_j(\zeta_j)>0$ for $\theta_i\in(1,\theta_i^*),\ \zeta_j\in(1,\zeta_j^*)$, respectively.

By choosing $\lambda=\min\{\theta_1^*,\theta_2^*,\dots,\theta_r^*,\zeta_1,\zeta_2,\dots,\zeta_s^*\}$, we obtain $\lambda>1$ and

$$\begin{aligned} \mathscr{A}_i(\lambda) \;=\;& \rho_i-\lambda(1-a_i)\rho_i \\ &-L_i^g\sum_{j=1}^{s}\xi_j(|d_{ji}|+|\alpha_{ji}|+|\beta_{ji}|)\lambda^{1+\sigma_{ji}} \\ \ge\;& 0,\ i=1,2,\dots,r, \end{aligned}$$

$$\begin{aligned} \mathscr{B}_j(\lambda) \;=\;& \xi_j-\lambda(1-b_j)\xi_j \\ &-L_j^f\sum_{i=1}^{r}\rho_i(|a_{ij}|+|c_{ij}|+|e_{ij}|)\lambda^{1+\tau_{ij}} \\ \ge\;& 0,\ j=1,2,\dots,s. \end{aligned}$$

Now define

$$\begin{cases} \mu_i(n)=\lambda^n|x_i(n)-x_i^*|,\ n\in N[-\sigma,+\infty),\ i=1,2,\dots,r, \\ \eta_j(n)=\lambda^n|y_j(n)-y_j^*|,\ n\in N[-\tau,+\infty),\ j=1,2,\dots,s. \end{cases} \tag{9}$$

Using (7),(8),(9), we derive that

$$\begin{cases} \mu_i(n+1) \;\le\; \lambda(1-a_i)\mu_i(n)+\sum\limits_{j=1}^{s}\lambda^{1+\tau_{ij}}|a_{ij}|L_j^f\eta_j(n) \\ \qquad\qquad +\sum\limits_{j=1}^{s}\lambda^{1+\tau_{ij}}(|c_{ij}|+|e_{ij}|)L_j^f\eta_j(n-\tau_{ij}), \\ \eta_j(n+1) \;\le\; \lambda(1-b_j)\eta_j(n)+\sum\limits_{i=1}^{r}\lambda^{1+\sigma_{ji}}|d_{ji}|L_i^g\mu_i(n) \\ \qquad\qquad +\sum\limits_{i=1}^{r}\lambda^{1+\sigma_{ji}}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g\mu_i(n-\sigma_{ji}), \end{cases} \tag{10}$$

where $n>0, n\ne n_k, k\in\mathbb{Z}^+, i=1,2,\dots,r, j=1,2,\dots,s$. Also,

$$\begin{cases} \Delta\mu_i(n_k) \;=\; (\lambda|1-\gamma_{ik}|-1)\mu_i(n_k),\ n\in N[-\sigma,+\infty), \\ \qquad\qquad i=1,2,\dots,r, \\ \Delta\eta_j(n_k) \;=\; (\lambda|1-\overline{\gamma_{jk}}|-1)\eta_j(n_k),\ n\in N[-\tau,+\infty), \\ \qquad\qquad j=1,2,\dots,s. \end{cases}$$

Define the Lyapunov function by

$$\begin{aligned} V(n) \;=\;& \sum_{i=1}^{r}\Bigg[\rho_i\mu_i(n)+\rho_i\sum_{j=1}^{s}\lambda^{1+\tau_{ij}}(|c_{ij}|+|e_{ij}|) \\ &\times L_j^f\Bigg(\sum_{l=n-\tau_{ij}}^{n-1}\eta_j(l)\Bigg)\Bigg] \\ &+\sum_{j=1}^{s}\Bigg[\xi_j\eta_j(n)+\xi_j\sum_{i=1}^{r}\lambda^{1+\sigma_{ji}}(|\alpha_{ji}|+|\beta_{ji}|) \end{aligned}$$

$$\times L_i^g\Big(\sum_{l=n-\sigma_{ji}}^{n-1}\mu_i(l)\Big)\Big] \tag{11}$$

and we note that $V(n) > 0$, for $n > 0$ and $V(0)$ is positive and finite. Calculating the difference of V along solutions of system (10), we obtain

$$\begin{aligned}
&\Delta V(n)\\
&= \sum_{i=1}^{r}\Big[\rho_i\mu_i(n+1)-\rho_i\mu_i(n)\\
&\quad +\rho_i\sum_{j=1}^{s}\lambda^{1+\tau_{ij}}(|c_{ij}|+|e_{ij}|)L_j^f(\eta_j(n)-\eta_j(n-\tau_{ij}))\Big]\\
&\quad +\sum_{j=1}^{s}\Big[\xi_j\eta_j(n+1)-\xi_j\eta_j(n)\\
&\quad +\xi_j\sum_{i=1}^{r}\lambda^{1+\sigma_{ji}}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g(\mu_i(n)-\mu_i(n-\sigma_{ji}))\Big]\\
&\le \sum_{i=1}^{r}\Big[\rho_i(\lambda(1-a_i)-1)\mu_i(n)+\rho_i\sum_{j=1}^{s}\lambda^{1+\tau_{ij}}|a_{ij}|L_j^f\eta_j(n)\\
&\quad +\rho_i\sum_{j=1}^{s}\lambda^{1+\tau_{ij}}(|c_{ij}|+|e_{ij}|)L_j^f\eta_j(n-\tau_{ij})\\
&\quad +\rho_i\sum_{j=1}^{s}\lambda^{1+\tau_{ij}}(|c_{ij}|+|e_{ij}|)L_j^f\eta_j(n)\\
&\quad -\rho_i\sum_{j=1}^{s}\lambda^{1+\tau_{ij}}(|c_{ij}|+|e_{ij}|)L_j^f\eta_j(n-\tau_{ij})\Big]\\
&\quad +\sum_{j=1}^{s}\Big[\xi_j(\lambda(1-b_j)-1)\eta_j(n)\\
&\quad +\xi_j\sum_{i=1}^{r}\lambda^{1+\sigma_{ji}}|d_{ji}|L_i^g\mu_i(n)\\
&\quad +\xi_j\sum_{i=1}^{r}\lambda^{1+\sigma_{ji}}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g\mu_i(n-\sigma_{ji})\\
&\quad +\xi_j\sum_{i=1}^{r}\lambda^{1+\sigma_{ji}}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g\mu_i(n)\\
&\quad -\xi_j\sum_{i=1}^{r}\lambda^{1+\sigma_{ji}}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g\mu_i(n-\sigma_{ji})\Big]\\
&= \sum_{i=1}^{r}\Big[\lambda(1-a_i)\rho_i+\sum_{j=1}^{s}\xi_j(|d_{ji}|+|\alpha_{ji}|\\
&\quad +|\beta_{ji}|)\lambda^{1+\sigma_{ji}}L_i^g-\rho_i\Big]\mu_i(n)\\
&\quad +\sum_{j=1}^{s}\Big[\lambda(1-b_j)\xi_j+\sum_{i=1}^{r}\rho_i(|a_{ij}|+|c_{ij}|\\
&\quad +|e_{ij}|)\lambda^{1+\tau_{ij}}L_j^f-\xi_j\Big]\eta_j(n)\\
&= -\sum_{i=1}^{r}\mathscr{A}_i(\lambda)\mu_i(n)-\sum_{j=1}^{s}\mathscr{B}_j(\lambda)\eta_j(n)\\
&\le 0,\ \text{for } n>0,\ n\ne n_k.
\end{aligned}$$

Also, from $a_i \le \gamma_{ik} \le 2-a_i,\ b_j \le \overline{\gamma_{jk}} \le 2-b_j$, we get

$$\begin{aligned}
\Lambda_i &= \rho_i(1-\lambda|1-\gamma_{ik}|)\\
&\quad -L_i^g\sum_{j=1}^{s}\xi_j\lambda^{1+\sigma_{ji}}(|d_{ji}|+|\alpha_{ji}|+|\beta_{ji}|)\\
&\ge \rho_i[1-\lambda(1-a_i)]\\
&\quad -L_i^g\sum_{j=1}^{s}\xi_j\lambda^{1+\sigma_{ji}}(|d_{ji}|+|\alpha_{ji}|+|\beta_{ji}|)>0,\\
&\quad i=1,2,\ldots,r,
\end{aligned}$$

$$\begin{aligned}
\Gamma_j &= \xi_j(1-\lambda|1-\overline{\gamma_{jk}}|)\\
&\quad -L_j^f\sum_{i=1}^{r}\rho_i\lambda^{1+\tau_{ij}}(|a_{ij}|+|c_{ij}|+|e_{ij}|)\\
&\ge \xi_j[1-\lambda(1-b_j)]\\
&\quad -L_j^f\sum_{i=1}^{r}\rho_i\lambda^{1+\tau_{ij}}(|a_{ij}|+|c_{ij}|+|e_{ij}|)>0,\\
&\quad j=1,2,\ldots,s.
\end{aligned}$$

Then

$$\begin{aligned}
&\Delta V(n_k)\\
&= \sum_{i=1}^{r}\Big[\rho_i\mu_i(n_k+1)-\rho_i\mu_i(n_k)\\
&\quad +\rho_i\sum_{j=1}^{s}\lambda^{1+\tau_{ij}}(|c_{ij}|+|e_{ij}|)L_j^f(\eta_j(n_k)-\eta_j(n_k-\tau_{ij}))\Big]\\
&\quad +\sum_{j=1}^{s}\Big[\xi_j\eta_j(n_k+1)-\xi_j\eta_j(n_k)\\
&\quad +\xi_j\sum_{i=1}^{r}\lambda^{1+\sigma_{ji}}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g(\mu_i(n_k)-\mu_i(n_k-\sigma_{ji}))\Big]\\
&\le \sum_{i=1}^{r}\Big[\rho_i(\lambda|1-\gamma_{ik}|-1)\mu_i(n_k)\\
&\quad +\rho_i\sum_{j=1}^{s}\lambda^{1+\tau_{ij}}(|c_{ij}|+|e_{ij}|)L_j^f\eta_j(n_k)\Big]\\
&\quad +\sum_{j=1}^{s}\Big[\xi_j(\lambda|1-\overline{\gamma_{jk}}|-1)\eta_j(n_k)\\
&\quad +\xi_j\sum_{i=1}^{r}\lambda^{1+\sigma_{ji}}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g\mu_i(n_k)\Big]\\
&\le \sum_{i=1}^{r}\Big[\rho_i(\lambda|1-\gamma_{ik}|-1)+\sum_{j=1}^{s}\xi_j\lambda^{1+\sigma_{ji}}(|d_{ji}|+|\alpha_{ji}|\\
&\quad +|\beta_{ji}|)L_i^g\Big]\mu_i(n_k)+\sum_{j=1}^{s}\Big[\xi_j(\lambda|1-\overline{\gamma_{jk}}|-1)\\
&\quad +\sum_{i=1}^{r}\rho_i\lambda^{1+\tau_{ij}}(|a_{ij}|+|c_{ij}|+|e_{ij}|)L_j^f\Big]\eta_j(n_k)\\
&= -\sum_{i=1}^{r}\Lambda_i\mu_i(n_k)-\sum_{j=1}^{s}\Gamma_j\eta_j(n_k)\\
&\le 0.
\end{aligned}$$

By (11), we have

$$\sum_{i=1}^{r}\rho_i\mu_i(n)+\sum_{j=1}^{s}\xi_j\eta_j(n)\le V(n)\le V(0)$$

and combining with (9), we obtain

$$\begin{aligned}
&\sum_{i=1}^{r}|x_i(n)-x_i^*|+\sum_{j=1}^{s}|y_j(n)-y_j^*|\\
=\;&\frac{1}{\lambda^n}\sum_{i=1}^{r}\Big[\rho_i|x_i(0)-x_i^*|+\rho_i\sum_{j=1}^{s}\lambda^{1+\tau_{ij}}(|c_{ij}|+|e_{ij}|)\\
&\times L_j^f\Big(\sum_{l=-\tau_{ij}}^{-1}|y_j(l)-y_j^*|\Big)\Big]\\
&+\frac{1}{\lambda^n}\sum_{j=1}^{s}\Big[\xi_j|y_j(0)-y_j^*|+\xi_j\sum_{i=1}^{r}\lambda^{1+\sigma_{ji}}(|\alpha_{ji}|+|\beta_{ji}|)\\
&\times L_i^g\Big(\sum_{l=-\sigma_{ji}}^{-1}|x_i(l)-x_i^*|\Big)\Big]\\
=\;&\frac{1}{\lambda^n}\sum_{i=1}^{r}\Big[\rho_i|x_i(0)-x_i^*|+\sum_{j=1}^{s}\xi_j\lambda^{1+\sigma_{ji}}(|\alpha_{ji}|+|\beta_{ji}|)\\
&\times L_i^g\Big(\sum_{l=-\sigma_{ji}}^{-1}|x_i(l)-x_i^*|\Big)\Big]\\
&+\frac{1}{\lambda^n}\sum_{j=1}^{s}\Big[\xi_j|y_j(0)-y_j^*|+\sum_{i=1}^{r}\rho_i\lambda^{1+\tau_{ij}}(|c_{ij}|+|e_{ij}|)\\
&\times L_j^f\Big(\sum_{l=-\tau_{ij}}^{-1}|y_j(l)-y_j^*|\Big)\Big]\\
\le\;&\frac{\kappa}{\lambda^n}\sum_{i=1}^{r}\Big[1+\sum_{j=1}^{s}\lambda^{1+\sigma_{ji}}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g\sigma_{ji}\Big]\\
&\times\sup_{l\in[-\sigma,0]}|x_i(l)-x_i^*|\\
&+\frac{\kappa}{\lambda^n}\sum_{j=1}^{s}\Big[1+\sum_{i=1}^{r}\lambda^{1+\tau_{ij}}(|c_{ij}|+|e_{ij}|)L_j^f\tau_{ij}\Big]\\
&\times\sup_{l\in[-\tau,0]}|y_j(l)-y_j^*|\\
\le\;&\frac{\kappa\varsigma}{\lambda^n}\Big(\sum_{i=1}^{r}\sup_{l\in[-\sigma,0]}|x_i(l)-x_i^*|+\sum_{j=1}^{s}\sup_{l\in[-\tau,0]}|y_j(l)-y_j^*|\Big),
\end{aligned}$$

where

$$\kappa=\frac{\max\limits_{1\le i\le r,\,1\le j\le s}\{\rho_i,\,\xi_j\}}{\min\limits_{1\le i\le r,\,1\le j\le s}\{\rho_i,\,\xi_j\}},$$

$$\begin{aligned}\varsigma=\max\Big\{&\max_{1\le i\le r}\Big\{1+\sum_{j=1}^{s}\lambda^{1+\sigma_{ji}}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g\sigma_{ji}\Big\},\\
&\max_{1\le j\le s}\Big\{1+\sum_{i=1}^{r}\lambda^{1+\tau_{ij}}(|c_{ij}|+|e_{ij}|)L_j^f\tau_{ij}\Big\}\Big\}.\end{aligned}$$

From above and Definition 1, we conclude that the equilibrium point $(x_1^*,\dots,x_r^*,y_1^*,\dots,y_s^*)^T\in\mathbb{R}^{r+s}$ of system (1) is globally exponentially stable. The proof is complete. ■

V. Some numerical examples

Example 1. *In system* (1), *let* $f_1(x)=f_2(x)=g_1(x)=g_2(x)=x, L_i^g=L_j^f=1,\sigma_{ji},\tau_{ij}\in(0,+\infty), i=1,2, j=1,2;$ $a_1=0.9$, $a_2=0.91$, $b_1=0.83$, $b_2=0.82$, $c_1=-0.91$, $c_2=0.13$, $d_1=-0.64$, $d_2=0.23$,

$$\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}=\begin{pmatrix}0.14&0.15\\0.16&0.11\end{pmatrix},\begin{pmatrix}c_{11}&c_{12}\\c_{21}&c_{22}\end{pmatrix}=\begin{pmatrix}0.11&0.12\\0.13&0.14\end{pmatrix},$$

$$\begin{pmatrix}e_{11}&e_{12}\\e_{21}&e_{22}\end{pmatrix}=\begin{pmatrix}0.12&0.09\\0.11&0.12\end{pmatrix},\begin{pmatrix}d_{11}&d_{12}\\d_{21}&d_{22}\end{pmatrix}=\begin{pmatrix}0.12&0.13\\0.15&0.16\end{pmatrix},$$

$$\begin{pmatrix}\alpha_{11}&\alpha_{12}\\\alpha_{21}&\alpha_{22}\end{pmatrix}=\begin{pmatrix}0.16&0.13\\0.14&0.15\end{pmatrix},\begin{pmatrix}\beta_{11}&\beta_{12}\\\beta_{21}&\beta_{22}\end{pmatrix}=\begin{pmatrix}0.14&0.13\\0.12&0.13\end{pmatrix},$$

$$\begin{pmatrix}m_{11}&m_{12}\\m_{21}&m_{22}\end{pmatrix}=\begin{pmatrix}0.11&0.13\\0.12&0.15\end{pmatrix},\begin{pmatrix}T_{11}&T_{12}\\T_{21}&T_{22}\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix},$$

$$\begin{pmatrix}H_{11}&H_{12}\\H_{21}&H_{22}\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix},$$

$$\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix}=\begin{pmatrix}0.12&0.09\\0.13&0.14\end{pmatrix},$$

$\gamma_{1k}=1+0.12\sin(1+k^2),\gamma_{2k}=1+0.11\sin k^2,\overline{\gamma_{1k}}=1+0.12\cos k^2,\overline{\gamma_{2k}}=1+0.13\cos(1+k^2),I_k(x_1(n_k))=-\gamma_{1k}(x_1(n_k)-4.0956),I_k(x_2(n_k))=-\gamma_{2k}(x_2(n_k)-4.1812),J_k(y_1(n_k))=-\overline{\gamma_{1k}}(y_1(n_k)-4.5204),J_k(y_2(n_k))=-\overline{\gamma_{2k}}(y_2(n_k)-4.7564)$. *Obviously,* $(H_1),(H_2)$ *is satisfied. From above, it is easy for us to calculate that*

$$a_1=0.9>L_1^g\sum_{j=1}^{2}(|d_{ji}|+|\alpha_{ji}|+|\beta_{ji}|)=0.83,$$

$$b_1=0.83>L_1^f\sum_{i=1}^{2}(|a_{ij}|+|c_{ij}|+|e_{ij}|)=0.77,$$

$$a_2=0.91>L_2^g\sum_{j=1}^{2}(|d_{ji}|+|\alpha_{ji}|+|\beta_{ji}|)=0.83,$$

$$b_2=0.82>L_2^f\sum_{i=1}^{2}(|a_{ij}|+|c_{ij}|+|e_{ij}|)=0.73,$$

$$\sum_{i=1}^{2}\Big[|1-\gamma_{ik}|+\sum_{j=1}^{2}(|\alpha_{ji}|+|\beta_{ji}|)L_i^g\Big]\le1.33<2=r,$$

$$\sum_{j=1}^{2}\Big[|1-\overline{\gamma_{jk}}|+\sum_{i=1}^{2}(|c_{ij}|+|e_{ij}|)L_j^f\Big]\le1.19<2=s.$$

Hence, $(H_3),(H_4)$ *are also satisfied, then according to Theorem 2, system* (1) *has a unique equilibrium point* $(4.0956,4.1812,4.5204,4.7564)$ *which is globally asymptotically stable.*

Example 2. *In system* (1), *let* $f_1(x)=f_2(x)=g_1(x)=g_2(x)=x, L_i^g=L_j^f=1,\sigma_{ji},\tau_{ij}\in(0,+\infty), i=1,2, j=1,2;$ $a_1=0.4$, $a_2=0.5$, $b_1=0.3$, $b_2=0.5$, $c_1=-1.34$, $c_2=-0.53$, $d_1=-1.63$, $d_2=-0.75$,

$$\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}=\begin{pmatrix}0.05&0.03\\0.02&0.04\end{pmatrix},\begin{pmatrix}c_{11}&c_{12}\\c_{21}&c_{22}\end{pmatrix}=\begin{pmatrix}0.01&0.03\\0.03&0.02\end{pmatrix},$$

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} 0.03 & 0.01 \\ 0.02 & 0.01 \end{pmatrix}, \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = \begin{pmatrix} 0.01 & 0.03 \\ 0.02 & 0.01 \end{pmatrix},$$

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 0.04 & 0.02 \\ 0.01 & 0.01 \end{pmatrix}, \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} 0.01 & 0.03 \\ 0.02 & 0.04 \end{pmatrix},$$

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 0.11 & 0.12 \\ 0.13 & 0.12 \end{pmatrix}, \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 0.13 & 0.11 \\ 0.12 & 0.11 \end{pmatrix},$$

$$\gamma_{1k} = 0.43e^{\frac{1}{k^2}}, \gamma_{2k} = 0.53e^{\frac{1}{k}}, \overline{\gamma_{1k}} = 0.32e^{\frac{1}{k^2+1}},$$

$$\overline{\gamma_{2k}} = 0.53e^{\frac{1}{k+1}}, I_k(x_1(n_k)) = -\gamma_{1k}(x_1(n_k) - 2.9295),$$

$$I_k(x_2(n_k)) = -\gamma_{2k}(x_2(n_k) - 1.7678),$$

$$J_k(y_1(n_k)) = -\overline{\gamma_{1k}}(y_1(n_k) - 2.5691),$$

$$J_k(y_2(n_k)) = -\overline{\gamma_{2k}}(y_2(n_k) - 1.3526).$$

Obviously, $(H_1), (H_2)$ *is satisfied. From above, it is easy for us to calculate that*

$$a_1 = 0.4 > L_1^g \sum_{j=1}^{2} (|d_{ji}| + |\alpha_{ji}| + |\beta_{ji}|) = 0.11,$$

$$b_1 = 0.5 > L_1^f \sum_{i=1}^{2} (|a_{ij}| + |c_{ij}| + |e_{ij}|) = 0.14,$$

$$a_2 = 0.3 > L_2^g \sum_{j=1}^{2} (|d_{ji}| + |\alpha_{ji}| + |\beta_{ji}|) = 0.15,$$

$$b_2 = 0.5 > L_2^f \sum_{i=1}^{2} (|a_{ij}| + |c_{ij}| + |e_{ij}|) = 0.14,$$

$$a_1 = 0.4 < 0.43 \le 0.43e^{\frac{1}{k^2}} \le 2 - a_1 = 1.6,$$

$$a_2 = 0.5 < 0.53 \le 0.53e^{\frac{1}{k}} \le 2 - a_2 = 1.5,$$

$$b_1 = 0.3 < 0.32 \le 0.32e^{\frac{1}{k^2+1}} \le 2 - b_1 = 1.7,$$

$$b_2 = 0.5 < 0.53 \le 0.53e^{\frac{1}{k+1}} \le 2 - b_2 = 1.5.$$

Hence, all the conditions in Theorem 3 are satisfied, then according to Theorem 3, system (1) *has a unique equilibrium point* $(2.9295, 1.7678, 2.5691, 1, 3526)$ *which is globally exponentially stable.*

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