An efficient iterative updating method for gyroscopic systems using measured modal data

Yongxin Yuan, Jiashang Jiang

Abstract—Updating gyroscopic systems using measured modal data can be mathematically formulated as following two problems. **Problem I**: Given $M_a \in \mathbf{R}^{n\times n}, \Lambda = \text{diag}\{\lambda_1, \cdots, \lambda_p\} \in \mathbf{C}^{p\times p}, X = [x_1, \cdots, x_p] \in \mathbf{C}^{n\times p}$, where Λ and X are closed under complex conjugation in the sense that $\lambda_{2j} = \bar{\lambda}_{2j-1} \in \mathbf{C}$, $x_{2j} = \bar{x}_{2j-1} \in \mathbf{C}^n$ for $j = 1, \cdots, l$, and $\lambda_k \in \mathbf{R}$, $x_k \in \mathbf{R}^n$ for $k = 2l+1, \cdots, p$, find a real symmetric matrix K and a real skew-symmetric matrix G (that is, $G^T = -G$) such that $M_a X\Lambda^2 + GX\Lambda + KX = 0$. **Problem II**: Given a real symmetric matrix $K_a \in \mathbf{R}^{n\times n}$ and a real skew-symmetric matrix G_a, find $(\hat{G}, \hat{K}) \in \mathbf{S_E}$ such that $\|\hat{G} - G_a\|^2 + \|\hat{K} - K_a\|^2 = \min_{(G,K)\in \mathbf{S_E}}(\|G - G_a\|^2 + \|K - K_a\|^2)$, where $\mathbf{S_E}$ is the solution set of Problem I and $\|\cdot\|$ is the Frobenius norm. This paper presents an iterative algorithm to solve Problem I and Problem II. By using the proposed iterative method, a solution of Problem I can be obtained within finite iteration steps in the absence of round errors, and the minimum Frobenius norm solution of Problem I can be obtained by choosing a special kind of initial matrix pair. Moreover, the optimal approximation solution $(\hat{G}, \hat{K})$ of Problem II can be obtained by finding the minimum Frobenius norm solution of a changed Problem I. Numerical results show that the presented method can be used to update finite element models to get better agreement between analytical and experimental modal parameters.

Keywords—model updating, iterative algorithm, gyroscopic system, partially prescribed spectral data, optimal approximation.

I. Introduction

Throughout this paper, we shall adopt the following notation. $\mathbf{C}^{m\times n}$ and $\mathbf{R}^{m\times n}$ denote the set of all $m \times n$ complex and real matrices, $\mathbf{SR}^{n\times n}$ and $\mathbf{SSR}^{n\times n}$ denote the set of all $n \times n$ symmetric and skew-symmetric matrices in $\mathbf{R}^{n\times n}$. A^T, $\text{tr}(A)$ and $R(A)$ stand for the transpose, the trace and the column space of the matrix A, respectively. I_n represents the identity matrix of order n. For $A, B \in \mathbf{R}^{m\times n}$, an inner product in $\mathbf{R}^{m\times n}$ is defined by $(A, B) = \text{tr}(B^T A)$, then $\mathbf{R}^{m\times n}$ is a Hilbert space. The matrix norm $\|\cdot\|$ induced by the inner product is the Frobenius norm. Given two matrices $A = [a_{ij}] \in \mathbf{R}^{m\times n}$ and $B \in \mathbf{R}^{p\times q}$, the Kronecker product of A and B is defined by $A \otimes B = [a_{ij}B] \in \mathbf{R}^{mp\times nq}$. Also, for an $m \times n$ matrix $A = [a_1, a_2, \cdots, a_n]$, where $a_i, i = 1, \cdots, n$, is the i-th column vector of A, the stretching function $\text{vec}(A)$ is defined as $\text{vec}(A) = [a_1^T, a_2^T, \cdots, a_n^T]^T$. Let A, B and X be some matrices with appropriate dimensions, then we have the following well-known identity [1].

$$\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X).$$

Yongxin Yuan: School of Mathematics and Physics, Jiangsu University of Science and Technology, Zhenjiang 212003, P R China. e-mail: yuanyx_703@163.com
Jiashang Jiang: School of Mathematics and Physics, Jiangsu University of Science and Technology, Zhenjiang 212003, P R China. e-mail: jjiashang@163.com

Consider an n degree-of-freedom vibratory system, attached to a rigid frame. Suppose that the frame rotates with a constant angular velocity. Then the free infinitesimal oscillations of the system about the frame are governed by the system of ordinary differential equations

$$M_a\ddot{q}(t) + G_a\dot{q}(t) + K_a q(t) = 0. \quad (1)$$

The system represented by (1) is called gyroscopic system. Eq.(1) is usually obtained by finite element techniques, and therefore, known as the finite element model. The vector $q(t)$ represents the generalized coordinates of the system. M_a, K_a and G_a are, respectively, called the analytical mass, stiffness, and gyroscopic matrices. In many practical applications, M_a is symmetric and positive definite ($M_a > 0$), K_a are real-valued symmetric, and G_a is always real-valued skew-symmetric (that is, $G_a^T = -G_a$). If a fundamental solution to (1) is represented by $q(t) = xe^{\lambda t}$, then the scalar λ and the vector x must solve the quadratic eigenvalue problem (QEP)

$$(\lambda^2 M_a + \lambda G_a + K_a)x = 0. \quad (2)$$

Complex numbers λ and nonzero complex vectors x for which this relation holds are, respectively, the eigenvalues and eigenvectors of the system. It is known that the equation of (2) has $2n$ finite eigenvalues over the complex field, provided that the leading matrix coefficient M_a is nonsingular. Note that the signification of the system (1) usually can be interpreted via the eigenvalues and eigenvectors of Eq.(2). Because of this connection, a lot of efforts have been devoted to the QEP in the literature. Many applications, properties and numerical methods for the QEP are surveyed in the thesis by Tisseur and Meerbergen [2].

In structural engineering analyses and designs, the finite element method has become the most widely used tool for numerical modeling. One important issue in this regard is how to refine the finite element model established such that it can accurately predict the dynamic characteristics of the structure under external disturbances. In the formulation of finite elements, we are concerned with the numerical errors resulting from the approximation involved. If the solutions obtained from a finite element analysis are close to the analytical ones, the finite element model established is said to be free of numerical errors. In engineering practice, however, we are concerned with the capability of a finite element model to reproduce the real structural behavior. Frequently, the results obtained by a finite element model deviate in some extent to those measured for the structure. Such errors are called the modeling errors, which can render a numerical modeling inapplicable. To narrow the gap between a finite element model and the real structure, the finite element model used should be refined or updated using the modal data (eigenvalues and eigenvectors) measured for the structure.

In the past decades, various techniques for updating mass and stiffness matrices for first-order linear systems (i.e., $G_a = 0$) using measured response data have been discussed by Baruch [3], Baruch and Bar-Itzhack [4], Berman [5], Berman and Nagy [6], Wei [7, 8, 9], Yang, Chen, Hsu [10], Yang and Chen [11], and Yuan [12]. For an account of the earlier methods, see the authoritative book by Friswell and Mottershead [13], an integral introduction of the basic theory of finite element model updating is given. For second-order structure systems, the theory and computation have been considered by Friswell, Inman and Pilkey [14, 15], Kuo, Lin and Xu [16], Chu, Chu, and Lin [17] and Yuan [18]. It is well known that the conservative gyroscopic systems are another important class of second-order systems. They correspond to spinning structures where the Coriolis inertia forces are taken into account. Examples of such systems include helicopter rotor blades and spin-stabilized satellites with flexible elastic appendages such as solar panels or antennas. The numerical methods for quadratic eigenvalue problems of gyroscopic systems can see [2, 19-24]. In view of in analytical model (1) for structure dynamics, the effect of Coriolis forces on structural dynamic systems is not well understood because it is purely dynamics property that can not be measured statically. Therefore, the correction of gyroscopic systems is very important. However, we observe that the iterative methods for model updating have received little attention in these years. In this paper we will develop an iterative method for the finite element model updating of conservative gyroscopic systems which can incorporate the measured model data into the finite element model to produce an adjusted finite element model on the gyroscopic and stiffness matrices that closely match the experimental modal data. The problem of updating gyroscopic and stiffness matrices simultaneously can be mathematically formulated as follows.

Problem I. Let $\Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_p\} \in \mathbf{C}^{p\times p}$ and $X = [x_1, \cdots, x_p] \in \mathbf{C}^{n\times p}$ be the measured eigenvalue and eigenvector matrices, where $p < n$ and both Λ and X are closed under complex conjugation in the sense that $\lambda_{2j} = \bar{\lambda}_{2j-1} \in \mathbf{C}$, $x_{2j} = \bar{x}_{2j-1} \in \mathbf{C}^n$ for $j = 1, \cdots, l$, and $\lambda_k \in \mathbf{R}$, $x_k \in \mathbf{R}^n$ for $k = 2l+1, \cdots, p$, find a real-valued symmetric matrix K and a real-valued skew-symmetric matrix G such that

$$M_a X\Lambda^2 + GX\Lambda + KX = 0. \tag{3}$$

It is well known that G_a and K_a are good approximations of G and K. The strategy for obtaining an improved model is to find G and K that satisfy (3) and deviate as little as possible from G_a and K_a. Thus, we should further solve the following optimal approximation problem.

Problem II. Let $\mathbf{S_E}$ be the solution set of Problem I. Find $(\hat{G}, \hat{K}) \in \mathbf{S_E}$ such that

$$\|\hat{G}-G_a\|^2+\|\hat{K}-K_a\|^2 = \min_{(G,K)\in\mathbf{S_E}} (\|G-G_a\|^2+\|K-K_a\|^2). \tag{4}$$

The paper is organized as follows. In Section 2, an efficient iterative method is presented to solve Problem I and Problem II. Then several properties of Algorithm 1 are proved. By using the proposed iterative method, a solution of Problem I can be obtained within finite iteration steps in the absence of round errors, and the minimum Frobenius norm solution of Problem I can be obtained by choosing a special kind of initial matrix pair. In addition, the optimal approximation solution of Problem II is provided by finding the minimum Frobenius norm solution of a new matrix equation. In Section 3, a numerical example is used to test the effectiveness of the proposed algorithm.

II. The solution of Problem I and Problem II

Let $0 < \beta_i = \mathrm{Im}(\lambda_i)$ (the imaginary part of the complex number λ_i), $y_i = \mathrm{Re}(x_i)$, $z_i = \mathrm{Im}(x_i)$ for $i = 1, 3, \cdots, 2l-1$, and

$$\tilde{\Lambda} = \mathrm{diag}\left\{ \begin{bmatrix} 0 & \beta_1 \\ -\beta_1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & \beta_{2l-1} \\ -\beta_{2l-1} & 0 \end{bmatrix}, \lambda_{2l+1}, \cdots, \lambda_p \right\} \in \mathbf{R}^{p\times p}, \tag{5}$$

$$\tilde{X} = [y_1, z_1, \cdots, y_{2l-1}, z_{2l-1}, x_{2l+1}, \cdots, x_p] \in \mathbf{R}^{n\times p}. \tag{6}$$

Then, the equation of (3) can be equivalently written as

$$G\tilde{X}\tilde{\Lambda} + K\tilde{X} = F, \quad \text{s. t.} \quad G \in \mathbf{SSR}^{n\times n}, \ K \in \mathbf{SR}^{n\times n}, \tag{7}$$

where $F = -M_a\tilde{X}\tilde{\Lambda}^2$.

By using the finite iterative methods established by Liang et al. [25], Sheng and Chen [26] and Dehghan and Hajarian [27], we can develop an iterative algorithm for solving Problem IP as follows.

Algorithm 1

S 1. Input matrices $\tilde{X} \in \mathbf{R}^{n\times p}$, $\tilde{\Lambda} \in \mathbf{R}^{p\times p}$ and $M_a \in \mathbf{SR}^{n\times n}$, and choose arbitrary $n \times n$ symmetric matrix K_1 and skew-symmetric matrix G_1.

S 2. Calculate
$R_1 = F - G_1\tilde{X}\tilde{\Lambda} - K_1\tilde{X}$;
$P_1 = \frac{1}{2}(R_1\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}R_1^T)$;
$Q_1 = \frac{1}{2}(R_1\tilde{X}^T + \tilde{X}R_1^T)$;
$s := 1$.

S 3. If $R_s = 0$, then stop and (G_s, K_s) is a solution to the equation of (7), that is, a solution of Problem I; elseif $R_s \neq 0$ but $P_s = 0$ and $Q_s = 0$, then stop and the equation of (7) has no solution; else $s := s + 1$.

S 4. Calculate
$G_s = G_{s-1} + \frac{\|R_{s-1}\|^2}{\|P_{s-1}\|^2+\|Q_{s-1}\|^2} P_{s-1}$;
$K_s = K_{s-1} + \frac{\|R_{s-1}\|^2}{\|P_{s-1}\|^2+\|Q_{s-1}\|^2} Q_{s-1}$;
$R_s = F - G_s\tilde{X}\tilde{\Lambda} - K_s\tilde{X}$
$= R_{s-1} - \frac{\|R_{s-1}\|^2}{\|P_{s-1}\|^2+\|Q_{s-1}\|^2}(P_{s-1}\tilde{X}\tilde{\Lambda} + Q_{s-1}\tilde{X})$;
$P_s = \frac{1}{2}(R_s\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}R_s^T) + \frac{\|R_s\|^2}{\|R_{s-1}\|^2}P_{s-1}$;
$Q_s = \frac{1}{2}(R_s\tilde{X}^T + \tilde{X}R_s^T) + \frac{\|R_s\|^2}{\|R_{s-1}\|^2}Q_{s-1}$.

S 5. Go to *S* 3.

From Algorithm 1, we can easily see that $G_s, P_s \in \mathbf{SSR}^{n\times n}$ and $K_s, Q_s \in \mathbf{SR}^{n\times n}$ for $s = 1, 2, \cdots$.

Definition 1 Assume that $Y, Z \in \mathbf{R}^{m\times n}$. The matrices Y, Z

are called orthogonal each other if $\mathrm{tr}(Y^T Z) = 0$.
About Algorithm 1, we present the following basic properties.

Lemma 1: The sequences $\{R_i\}, \{P_i\}$ and $\{Q_i\}$ generated by Algorithm 1 satisfy

$$\begin{array}{l}\mathrm{tr}(R_j^T R_i) = 0, \quad and \quad \mathrm{tr}(P_j^T P_i) + \mathrm{tr}(Q_j^T Q_i) = 0 \\ for \quad i, j = 1, 2, \cdots, s, \ i \neq j.\end{array} \tag{8}$$

Proof. Since $\mathrm{tr}(R_j^T R_i) = \mathrm{tr}(R_i^T R_j), \mathrm{tr}(P_j^T P_i) = \mathrm{tr}(P_i^T P_j)$ and $\mathrm{tr}(Q_j^T Q_i) = \mathrm{tr}(Q_i^T Q_j)$, then we need only to show that $\mathrm{tr}(R_j^T R_i) = 0, \quad and \quad \mathrm{tr}(P_j^T P_i) + \mathrm{tr}(Q_j^T Q_i) = 0 \ for \ 1 \leq i < j \leq s$. We use the mathematic induction to prove this conclusion, and we do it in two steps.
We first show that

$$\begin{array}{l}\mathrm{tr}(R_{i+1}^T R_i) = 0, \quad and \quad \mathrm{tr}(P_{i+1}^T P_i) + \mathrm{tr}(Q_{i+1}^T Q_i) = 0 \\ for \quad i = 1, 2, \cdots, s.\end{array} \tag{9}$$

For $i = 1$, by Algorithm 1 and noting that $P_1 \in \mathbf{SSR}^{n\times n}$ and $Q_1 \in \mathbf{SR}^{n\times n}$, we have

$$\begin{aligned}
& \mathrm{tr}(R_2^T R_1) \\
= \ & \mathrm{tr}((R_1 - \xi_1(P_1\tilde{X}\tilde{\Lambda} + Q_1\tilde{X}))^T R_1) \\
= \ & \mathrm{tr}(R_1^T R_1) - \xi_1 \mathrm{tr}(\tilde{\Lambda}^T\tilde{X}^T P_1^T R_1 + \tilde{X}^T Q_1^T R_1) \\
= \ & \|R_1\|^2 - \frac{1}{2}\xi_1 \mathrm{tr}(\tilde{\Lambda}^T\tilde{X}^T P_1^T R_1 \\
+ \ & R_1^T P_1\tilde{X}\tilde{\Lambda} + \tilde{X}^T Q_1^T R_1 + R_1^T Q_1\tilde{X}) \\
= \ & \|R_1\|^2 - \frac{1}{2}\xi_1 \mathrm{tr}(P_1^T R_1\tilde{\Lambda}^T\tilde{X}^T \\
- \ & P_1^T\tilde{X}\tilde{\Lambda}R_1^T + Q_1^T R_1\tilde{X}^T + Q_1^T\tilde{X}R_1^T) \\
= \ & \|R_1\|^2 - \xi_1 \mathrm{tr}(P_1^T P_1 + Q_1^T Q_1) = 0,
\end{aligned}$$

where $\xi_1 = \frac{\|R_1\|^2}{\|P_1\|^2 + \|Q_1\|^2}$.
Applying the proved result $\mathrm{tr}(R_2^T R_1) = 0$, we get

$$\begin{aligned}
& \mathrm{tr}(P_2^T P_1) + \mathrm{tr}(Q_2^T Q_1) \\
= \ & \frac{1}{2}\mathrm{tr}((R_2\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}R_2^T)^T P_1) + \frac{\|R_2\|^2}{\|R_1\|^2}\|P_1\|^2 \\
+ \ & \frac{1}{2}\mathrm{tr}((R_2\tilde{X}^T + \tilde{X}R_2^T)Q_1) + \frac{\|R_2\|^2}{\|R_1\|^2}\|Q_1\|^2 \\
= \ & \frac{1}{2}\mathrm{tr}(R_2(P_1\tilde{X}\tilde{\Lambda} + Q_1\tilde{X})^T + (P_1\tilde{X}\tilde{\Lambda} + Q_1\tilde{X})R_2^T) \\
+ \ & \frac{\|R_2\|^2}{\|R_1\|^2}\|P_1\|^2 + \frac{\|R_2\|^2}{\|R_1\|^2}\|Q_1\|^2 \\
= \ & \frac{1}{2}\frac{\|P_1\|^2 + \|Q_1\|^2}{\|R_1\|^2}\mathrm{tr}((R_2(R_1 - R_2)^T \\
+ \ & (R_1 - R_2)R_2^T) + \frac{\|R_2\|^2}{\|R_1\|^2}\|P_1\|^2 + \frac{\|R_2\|^2}{\|R_1\|^2}\|Q_1\|^2 \\
= \ & 0.
\end{aligned}$$

Suppose that (9) holds for $i = t - 1$. For $i = t$, we have

$$\begin{aligned}
& \mathrm{tr}(R_{t+1}^T R_t) \\
= \ & \mathrm{tr}((R_t - \xi_t(P_t\tilde{X}\tilde{\Lambda} + Q_t\tilde{X}))^T R_t) \\
= \ & \mathrm{tr}(R_t^T R_t) - \xi_t \mathrm{tr}(\tilde{\Lambda}^T\tilde{X}^T P_t^T R_t + \tilde{X}^T Q_t^T R_t) \\
= \ & \|R_t\|^2 - \frac{1}{2}\xi_t \mathrm{tr}(\tilde{\Lambda}^T\tilde{X}^T P_t^T R_t + R_t^T P_t\tilde{X}\tilde{\Lambda} \\
+ \ & \tilde{X}^T Q_t^T R_t + R_t^T Q_t\tilde{X}) \\
= \ & \|R_t\|^2 - \frac{1}{2}\xi_t \mathrm{tr}(P_t^T R_t\tilde{\Lambda}^T\tilde{X}^T - P_t^T\tilde{X}\tilde{\Lambda}R_t^T \\
+ \ & Q_t^T R_t\tilde{X}^T + Q_t^T\tilde{X}R_t^T) \\
= \ & \|R_t\|^2 - \xi_t \mathrm{tr}(P_t^T(P_t - \frac{\|R_t\|^2}{\|R_{t-1}\|^2}P_{t-1}) \\
+ \ & Q_t^T(Q_t - \frac{\|R_t\|^2}{\|R_{t-1}\|^2}Q_{t-1})) \\
= \ & \|R_t\|^2 - \xi_t \mathrm{tr}(P_t^T P_t + Q_t^T Q_t) = 0,
\end{aligned}$$

where $\xi_t = \frac{\|R_t\|^2}{\|P_t\|^2 + \|Q_t\|^2}$.

$$\begin{aligned}
& \mathrm{tr}(P_{t+1}^T P_t) + \mathrm{tr}(Q_{t+1}^T Q_t) \\
= \ & \frac{1}{2}\mathrm{tr}((R_{t+1}\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}R_{t+1}^T)^T P_t) + \frac{\|R_{t+1}\|^2}{\|R_t\|^2}\|P_t\|^2 \\
+ \ & \frac{1}{2}\mathrm{tr}((R_{t+1}\tilde{X}^T + \tilde{X}R_{t+1}^T)Q_t) + \frac{\|R_{t+1}\|^2}{\|R_t\|^2}\|Q_t\|^2 \\
= \ & \frac{1}{2}\mathrm{tr}(R_{t+1}(P_t\tilde{X}\tilde{\Lambda} + Q_t\tilde{X})^T + (P_t\tilde{X}\tilde{\Lambda} + Q_t\tilde{X})R_{t+1}^T) \\
+ \ & \frac{\|R_{t+1}\|^2}{\|R_t\|^2}\|P_t\|^2 + \frac{\|R_{t+1}\|^2}{\|R_t\|^2}\|Q_t\|^2 \\
= \ & \frac{1}{2}\frac{\|P_t\|^2 + \|Q_t\|^2}{\|R_t\|^2}\mathrm{tr}((R_{t+1}(R_t - R_{t+1})^T \\
+ \ & (R_t - R_{t+1})R_{t+1}^T) + \frac{\|R_{t+1}\|^2}{\|R_t\|^2}\|P_t\|^2 \\
+ \ & \frac{\|R_{t+1}\|^2}{\|R_t\|^2}\|Q_t\|^2 = 0.
\end{aligned}$$

Therefore, (9) holds for $i = t$. By the principle of induction, we know (9) holds for all i.

Next, assume that $\mathrm{tr}(R_{i+d}^T R_i) = 0$, and $\mathrm{tr}(P_{i+d}^T P_i) + \mathrm{tr}(Q_{i+d}^T Q_i) = 0$ for $1 \leq i \leq s$ and $1 < d < s$. We will prove

$$\mathrm{tr}(R_{i+d+1}^T R_i) = 0, \text{ and } \mathrm{tr}(P_{i+d+1}^T P_i) + \mathrm{tr}(Q_{i+d+1}^T Q_i) = 0.$$

$$\begin{aligned}
& \mathrm{tr}(R_{i+d+1}^T R_i) \\
= \ & \mathrm{tr}((R_{i+d} - \delta(P_{i+d}\tilde{X}\tilde{\Lambda} + Q_{i+d}\tilde{X}))^T R_i) \\
= \ & -\delta \ \mathrm{tr}(\tilde{\Lambda}^T\tilde{X}^T P_{i+d}^T R_i + \tilde{X}^T Q_{i+d}^T R_i) \\
= \ & -\frac{1}{2}\delta \ \mathrm{tr}(\tilde{\Lambda}^T\tilde{X}^T P_{i+d}^T R_i \\
+ \ & R_i^T P_{i+d}\tilde{X}\tilde{\Lambda} + \tilde{X}^T Q_{i+d}^T R_i + R_i^T Q_{i+d}\tilde{X}) \\
= \ & -\frac{1}{2}\delta \ \mathrm{tr}(P_{i+d}^T R_i\tilde{\Lambda}^T\tilde{X}^T - P_{i+d}^T\tilde{X}\tilde{\Lambda}R_i^T \\
+ \ & Q_{i+d}^T R_i\tilde{X}^T + Q_{i+d}^T\tilde{X}R_i^T) \\
= \ & -\delta \ \mathrm{tr}(P_{i+d}^T(P_i - \frac{\|R_i\|^2}{\|R_{i-1}\|^2}P_{i-1}) \\
+ \ & Q_{i+d}^T(Q_i - \frac{\|R_i\|^2}{\|R_{i-1}\|^2}Q_{i-1})) = 0,
\end{aligned}$$

where $\delta = \frac{\|R_{i+d}\|^2}{\|P_{i+d}\|^2 + \|Q_{i+d}\|^2}$.
From the above results, we have $\mathrm{tr}(R_{i+d+1}^T R_i) = 0$ and

$\mathrm{tr}(R_{i+d+1}^T R_{i+1}) = 0$. Hence we can get

$$\begin{aligned} & \mathrm{tr}(P_{i+d+1}^T P_i) + \mathrm{tr}(Q_{i+d+1}^T Q_i) \\ = & \frac{1}{2}\mathrm{tr}((R_{i+d+1}\tilde{\Lambda}^T \tilde{X}^T - \tilde{X}\tilde{\Lambda}R_{i+d+1}^T)^T P_i) \\ + & \frac{1}{2}\mathrm{tr}((R_{i+d+1}\tilde{X}^T + \tilde{X}R_{i+d+1}^T)Q_i) \\ = & \frac{1}{2}\mathrm{tr}(R_{i+d+1}(P_i\tilde{X}\tilde{\Lambda} + Q_i\tilde{X})^T \\ + & (P_i\tilde{X}\tilde{\Lambda} + Q_i\tilde{X})R_{i+d+1}^T) \\ = & \frac{1}{2}\eta\ \mathrm{tr}((R_{i+d+1}(R_i - R_{i+1})^T \\ + & (R_i - R_{i+1})R_{i+d+1}^T) = 0, \end{aligned}$$

where $\eta = \frac{\|P_i\|^2 + \|Q_i\|^2}{\|R_i\|^2}$.

Thus the conclusion (8) holds by the principle of induction. The proof is completed.

Lemma 2: Let Problem I be consistent, and (G^*, K^*) be an arbitrary solution of Problem I. Then, for any initial matrix pair (G_1, K_1) with $G_1 \in \mathbf{SSR}^{n\times n}$ and $K_1 \in \mathbf{SR}^{n\times n}$, we have

$$\begin{aligned} & \mathrm{tr}((G^* - G_i)^T P_i) + \mathrm{tr}((K^* - K_i)^T Q_i) = \|R_i\|^2 \\ & for \quad i = 1, 2, \cdots, \end{aligned} \qquad (10)$$

where the sequences $\{G_i\}, \{K_i\}, \{R_i\}, \{P_i\}$ and $\{Q_i\}$ are generated by Algorithm 1.

Proof. We prove the conclusion by induction. For $i = 1$, we have

$$\begin{aligned} & \mathrm{tr}((G^* - G_1)^T P_1) + \mathrm{tr}((K^* - K_1)^T Q_1) \\ = & \frac{1}{2}\mathrm{tr}((G^* - G_1)^T (R_1\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}R_1^T)) \\ + & \frac{1}{2}\mathrm{tr}((K^* - K_1)^T (R_1\tilde{X}^T + \tilde{X}R_1^T)) \\ = & \frac{1}{2}\mathrm{tr}(FR_1^T - G_1\tilde{X}\tilde{\Lambda}R_1^T - K_1\tilde{X}R_1^T) \\ + & \frac{1}{2}\mathrm{tr}(F^T R_1 - \tilde{\Lambda}^T\tilde{X}^T G_1^T R_1 - \tilde{X}^T K_1^T R_1) \\ = & \frac{1}{2}\mathrm{tr}(R_1 R_1^T) + \frac{1}{2}\mathrm{tr}(R_1^T R_1) \\ = & \|R_1\|^2. \end{aligned}$$

Now assume the conclusion (10) holds for $1 \le i \le t-1$. Then we can get

$$\begin{aligned} & \mathrm{tr}((G^* - G_t)^T P_t) + \mathrm{tr}((K^* - K_t)^T Q_t) \\ = & \mathrm{tr}((G^* - G_{t-1} - \frac{\|R_{t-1}\|^2}{\|P_{t-1}\|^2 + \|Q_{t-1}\|^2}P_{t-1})^T P_t) \\ + & \mathrm{tr}((K^* - K_{t-1} - \frac{\|R_{t-1}\|^2}{\|P_{t-1}\|^2 + \|Q_{t-1}\|^2}Q_{t-1})^T Q_t) \\ = & \mathrm{tr}((G^* - G_{t-1})^T P_t) + \mathrm{tr}((K^* - K_{t-1})^T Q_t) \\ = & \frac{1}{2}\mathrm{tr}((G^* - G_{t-1})^T (R_t\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}R_t^T)) \\ + & \frac{\|R_t\|^2}{\|R_{t-1}\|^2}\mathrm{tr}((G^* - G_{t-1})^T P_{t-1}) \\ + & \frac{1}{2}\mathrm{tr}((K^* - K_{t-1})^T (R_t\tilde{X}^T + \tilde{X}R_t^T)) \\ + & \frac{\|R_t\|^2}{\|R_{t-1}\|^2}\mathrm{tr}((K^* - K_{t-1})^T Q_{t-1}) \\ = & \frac{1}{2}\mathrm{tr}((G^* - G_{t-1})^T (R_t\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}R_t^T)) \\ + & \frac{1}{2}\mathrm{tr}((K^* - K_{t-1})^T (R_t\tilde{X}^T + \tilde{X}R_t^T)) + \|R_t\|^2 \\ = & \frac{1}{2}\mathrm{tr}(F^T R_t - \tilde{\Lambda}^T\tilde{X}^T G_{t-1}^T R_t - \tilde{X}^T K_{t-1}^T R_t) \\ + & \frac{1}{2}\mathrm{tr}(FR_t^T - G_{t-1}\tilde{X}\tilde{\Lambda}R_t^T - K_{t-1}\tilde{X}R_t^T) + \|R_t\|^2 \\ = & \frac{1}{2}\mathrm{tr}(R_{t-1}^T R_t) + \frac{1}{2}\mathrm{tr}(R_{t-1}R_t^T) + \|R_t\|^2 \\ = & \|R_t\|^2. \end{aligned}$$

Thus we complete the proof of Lemma 2 by the principle of induction.

From Lemma 2, we can easily see that if there exists a positive number l such that $P_l = 0$ and $Q_l = 0$ but $R_l \neq 0$, then the equation of (7) has no solution. Hence, the solvability of Eq.(7) can be determined automatically by Algorithm 1.

Theorem 1: Assume that Problem I is consistent. Then for any arbitrary initial matrix pair (G_1, K_1) with $G_1 \in \mathbf{SSR}^{n\times n}$ and $K_1 \in \mathbf{SR}^{n\times n}$, a solution of Problem I can be obtained with finite iteration steps in the absence of roundoff errors.

Proof. Assume that $R_l \neq 0,\ l = 1, 2, \cdots, np$. From Lemma 2, we know $P_l \neq 0$ or $Q_l \neq 0$. Then we can calculate R_{np+1} and (G_{np+1}, K_{np+1}) by Algorithm 1. From Lemma 1, we have

$$\mathrm{tr}(R_{np+1}^T R_t) = 0,\ t = 1, 2, \cdots, np,$$

and

$$\mathrm{tr}(R_j^T R_i) = 0, \quad i, j = 1, 2, \cdots, np,\ i \neq j.$$

Therefore, $\{R_1, R_2, \cdots, R_{np}\}$ forms an orthogonal basis of the real-valued matrix space $\mathbf{R}^{n\times p}$, which implies that $R_{np+1} = 0$, that is, (G_{np+1}, K_{np+1}) is a solution of Problem I.

Lemma 3: The equation of (7) has a solution (G, K) with $G \in \mathbf{SSR}^{n\times n}$ and $K \in \mathbf{SR}^{n\times n}$ if and only if the matrix equations

$$\begin{aligned} G\tilde{X}\tilde{\Lambda} + K\tilde{X} = F, \\ -\tilde{\Lambda}^T\tilde{X}^T G + \tilde{X}^T K = F^T, \end{aligned} \qquad (11)$$

are consistent.

Proof. If the equation of (7) has a solution (G^*, K^*) with $G^* \in \mathbf{SSR}^{n\times n}$ and $K^* \in \mathbf{SR}^{n\times n}$, then $D^*\tilde{X}\tilde{\Lambda} + K^*\tilde{X} = F$, and $(G^*\tilde{X}\tilde{\Lambda} + K^*\tilde{X})^T = -\tilde{\Lambda}^T\tilde{X}^T G^* + \tilde{X}^T K^* = F^T$. That is to say, (G^*, K^*) is a solution of (11).

Conversely, if the matrix equations of (11) has a solution, say, $G = U,\ K = V$. Let $G^* = \frac{1}{2}(U - U^T),\ K^* = \frac{1}{2}(V + V^T)$, then G^* is skew-symmetric matrix and K^* is a symmetric matrix, and

$$\begin{aligned} & G^*\tilde{X}\tilde{\Lambda} + K^*\tilde{X} = \tfrac{1}{2}(U\tilde{X}\tilde{\Lambda} + V\tilde{X}) + \tfrac{1}{2}(-U^T\tilde{X}\tilde{\Lambda} + V^T\tilde{X}) \\ & = \tfrac{1}{2}F + \tfrac{1}{2}(F^T)^T = F. \end{aligned}$$

Hence, (G^*, K^*) is a solution of (7).

The following lemma is taken from [28].

Lemma 4: Suppose that the consistent system of linear equations $Ax = b$ has a solution $x \in R(A^T)$, then x is the unique minimum Frobenius norm solution of the system of linear equations.

Using the Kronecker product and the stretching function, we know that the equations of (11) are equivalent to

$$\begin{bmatrix} \tilde{\Lambda}^T\tilde{X}^T\otimes I_n & \tilde{X}^T\otimes I_n \\ I_n\otimes(-\tilde{\Lambda}^T\tilde{X}^T) & I_n\otimes\tilde{X}^T \end{bmatrix}\begin{bmatrix} \mathrm{vec}(G) \\ \mathrm{vec}(K) \end{bmatrix} = \begin{bmatrix} \mathrm{vec}(F) \\ \mathrm{vec}(F^T) \end{bmatrix}.$$

Assume that $H\in\mathbf{R}^{n\times n}$ is an arbitrary matrix, then we have

$$\begin{aligned} &\begin{bmatrix} \mathrm{vec}(H\tilde{\Lambda}^T\tilde{X}^T-\tilde{X}\tilde{\Lambda}H^T) \\ \mathrm{vec}(H\tilde{X}^T+\tilde{X}H^T) \end{bmatrix} \\ = &\begin{bmatrix} \tilde{X}\tilde{\Lambda}\otimes I_n & I_n\otimes(-\tilde{X}\tilde{\Lambda}) \\ \tilde{X}\otimes I_n & I_n\otimes\tilde{X} \end{bmatrix}\begin{bmatrix} \mathrm{vec}(H) \\ \mathrm{vec}(H^T) \end{bmatrix} \\ = &\begin{bmatrix} \tilde{\Lambda}^T\tilde{X}^T\otimes I_n & \tilde{X}^T\otimes I_n \\ I_n\otimes(-\tilde{\Lambda}^T\tilde{X}^T) & I_n\otimes\tilde{X}^T \end{bmatrix}^T\begin{bmatrix} \mathrm{vec}(H) \\ \mathrm{vec}(H^T) \end{bmatrix} \\ \in\ & R\left(\begin{bmatrix} \tilde{\Lambda}^T\tilde{X}^T\otimes I_n & \tilde{X}^T\otimes I_n \\ I_n\otimes(-\tilde{\Lambda}^T\tilde{X}^T) & I_n\otimes\tilde{X}^T \end{bmatrix}^T\right). \end{aligned}$$

It is obvious that if we choose

$$G_1 = H\tilde{\Lambda}^T\tilde{X}^T-\tilde{X}\tilde{\Lambda}H^T,\quad K_1 = H\tilde{X}^T+\tilde{X}H^T, \tag{12}$$

then all G_s and K_s generated by Algorithm 1 satisfy

$$\begin{bmatrix} \mathrm{vec}(G_s) \\ \mathrm{vec}(K_s) \end{bmatrix} \in R\left(\begin{bmatrix} \tilde{\Lambda}^T\tilde{X}^T\otimes I_n & \tilde{X}^T\otimes I_n \\ I_n\otimes(-\tilde{\Lambda}^T\tilde{X}^T) & I_n\otimes\tilde{X}^T \end{bmatrix}^T\right).$$

It follows from Lemma 4 that if we choose a initial matrix pair by (12), where H is an arbitrary matrix, then a solution (G^*, K^*) obtained by Algorithm 1 is the minimum Frobenius norm solution of Problem I. In summary of above discussion, we have proved the following result.

Theorem 2: Suppose that Problem I is consistent. If we choose the initial matrices by (12), where H is an arbitrary matrix, or especially, $G_1 = 0$ and $K_1 = 0$, then we can obtain the unique minimum Frobenius norm solution of Problem I within finite iterative steps.

Now we show that the solution of Problem II can be derived by finding the minimum norm solution of a new matrix equation. Assume that Problem I is consistent. Obviously the solution set $\mathbf{S_E}$ of Problem I is nonempty, then for a given matrix pair (G_a, K_a), we have
$G\tilde{X}\tilde{\Lambda}+K\tilde{X} = -M_a\tilde{X}\tilde{\Lambda}^2 \Leftrightarrow (G-G_a)\tilde{X}\tilde{\Lambda}+(K-K_a)\tilde{X} = -M_a\tilde{X}\tilde{\Lambda}^2-G_a\tilde{X}\tilde{\Lambda}-K_a\tilde{X}$.
Let

$$\tilde{G} = G-G_a,\ \ \tilde{K} = K-K_a,\ \ \tilde{F} = -M_a\tilde{X}\tilde{\Lambda}^2-G_a\tilde{X}\tilde{\Lambda}-K_a\tilde{X},$$

then the matrix approximation Problem II is equivalent to finding the minimum Frobenius norm solution of the matrix equation

$$\tilde{G}\tilde{X}\tilde{\Lambda}+\tilde{K}\tilde{X} = \tilde{F},\ \text{ s. t. } \tilde{G}\in\mathbf{SSR}^{n\times n},\ \tilde{K}\in\mathbf{SR}^{n\times n}. \tag{13}$$

Applying Algorithm 1, and taking the initial matrix pair by $\tilde{G}_1 = H\tilde{\Lambda}^T\tilde{X}^T-\tilde{X}\tilde{\Lambda}H^T,\ \tilde{K}_1 = H\tilde{X}^T+\tilde{X}H^T$, where H is an arbitrary matrix, or especially, $\tilde{G}_1 = 0$ and $\tilde{K}_1 = 0$, we can obtain the minimum Frobenius norm solution $(\tilde{G}^*, \tilde{K}^*)$ of (13). Once the above matrix pair $(\tilde{G}^*, \tilde{K}^*)$ is obtained, the solution of the matrix optimal approximation Problem II can be computed. In this case, can be expressed as

$$\hat{G} = G_a+\tilde{G}^*,\quad \hat{K} = K_a+\tilde{K}^*. \tag{14}$$

III. A NUMERICAL EXAMPLE

In this section, we will give a numerical example to illustrate our results. All the tests are performed using MATLAB 6.5. Because of the influence of the error of calculation, the iteration will not stop within finite steps. Hence, we regard (G_s, K_s) as a solution of the considered problem if the corresponding residue satisfies $\|R_s\| \le 1.0e-005$.

Example 1. Consider a 7-DOF system modelled analytically with mass, gyroscopic, and stiffness matrices given by

$$M_a = 0.03\times\begin{bmatrix} 52 & 22 & 18 & -13 & 0 & 0 & 0 \\ 22 & 12 & 13 & -9 & 0 & 0 & 0 \\ 18 & 13 & 104 & 0 & 18 & -13 & 0 \\ -13 & -9 & 0 & 24 & 13 & -9 & 0 \\ 0 & 0 & 18 & 13 & 104 & 0 & 18 \\ 0 & 0 & -13 & -9 & 0 & 24 & 13 \\ 0 & 0 & 0 & 0 & 18 & 13 & 104 \end{bmatrix},$$

$$G_a = \begin{bmatrix} 0 & 15.06 & -9.06 & 14.01 & 0 & 0 & 0 \\ -15.06 & 0 & -14.01 & 14.13 & 0 & 0 & 0 \\ 9.06 & 14.01 & 0 & 0 & -9.06 & 14.01 & 0 \\ -14.01 & -14.13 & 0 & 0 & -14.01 & 14.13 & 0 \\ 0 & 0 & 9.06 & 14.01 & 0 & 0 & -9.06 \\ 0 & 0 & -14.01 & -14.13 & 0 & 0 & -14.01 \\ 0 & 0 & 0 & 0 & 9.06 & 14.01 & 0 \end{bmatrix}$$

and

$$K_a = 600\times\begin{bmatrix} 2 & 3 & -2 & 3 & 0 & 0 & 0 \\ 3 & 6 & -3 & 3 & 0 & 0 & 0 \\ -2 & -3 & 4 & 0 & -2 & 3 & 0 \\ 3 & 3 & 0 & 12 & -3 & 3 & 0 \\ 0 & 0 & -2 & -3 & 4 & 0 & -2 \\ 0 & 0 & 3 & 3 & 0 & 12 & -3 \\ 0 & 0 & 0 & 0 & -2 & -3 & 4 \end{bmatrix}.$$

The measured eigenvalue and eigenvector matrices Λ and X are given by

$$\Lambda = \mathrm{diag}\{10.503i, -10.503i, 1.9013i, -1.9013i\}$$

and

$$X = \begin{bmatrix} 0.0751+0.6464i & 0.0751-0.6464i & -0.8475-0.0112i & -0.8475+0.0112i \\ -0.0279-0.3178i & -0.0279+0.3178i & 0.1286-0.0194i & 0.1286+0.0194i \\ 0.0659-0.2447i & 0.0659+0.2447i & -0.4680-0.0299i & -0.4680+0.0299i \\ -0.0028-0.2284i & -0.0028+0.2284i & 0.1216-0.0010i & 0.1216+0.0010i \\ 0.0126-0.5021i & 0.0126+0.5021i & -0.1502-0.0158i & -0.1502+0.0158i \\ -0.0060+0.0579i & -0.0060-0.0579i & 0.0837+0.0056i & 0.0837-0.0056i \\ -0.0157-0.2507i & -0.0157+0.2507i & -0.0125-0.0036i & -0.0125+0.0036i \end{bmatrix}.$$

Choosing initial iterative matrix pair $\tilde{G}_1 = 0$, $\tilde{K}_1 = 0$. By Algorithm 1, we get the minimum Frobenius norm solution $(\tilde{G}^*, \tilde{K}^*)$ of Eq.(13) as follows.

$$\tilde{G}^* = \tilde{G}_{26} = \begin{bmatrix} 0.0000 & -1.0790 & -0.2173 & -0.5109 & 0.4572 & -0.1521 & 0.6171 \\ 1.0790 & 0.0000 & -1.4091 & -0.5108 & 1.1087 & -0.5532 & 1.9058 \\ 0.2173 & 1.4091 & 0.0000 & -0.9276 & 0.5932 & -0.4885 & 0.7779 \\ 0.5109 & 0.5108 & 0.9276 & -0.0000 & 0.1741 & -0.1056 & -0.2554 \\ -0.4572 & -1.1087 & -0.5932 & -0.1741 & -0.0000 & -0.3502 & -0.0273 \\ 0.1521 & 0.5532 & 0.4885 & 0.1056 & 0.3502 & -0.0000 & 0.1637 \\ -0.6171 & -1.9058 & -0.7779 & 0.2554 & 0.0273 & -0.1637 & -0.0000 \end{bmatrix},$$

$$\tilde{K}^* = \tilde{K}_{26} = \begin{bmatrix} 6.1356 & 29.6371 & 2.5124 & -7.3514 & -19.3481 & -0.8994 & -11.3819 \\ 29.6371 & 85.3847 & -37.4768 & -31.5535 & -9.1961 & -8.5444 & 33.4304 \\ 2.5124 & -37.4768 & -25.4059 & 1.9575 & 35.9010 & -0.8869 & 38.1903 \\ -7.3514 & -31.5535 & 1.9575 & 8.7103 & 15.3619 & 1.6029 & 5.0016 \\ -19.3481 & -9.1961 & 35.9010 & 15.3619 & 5.6428 & 1.7733 & -19.7534 \\ -0.8994 & -8.5444 & -0.8869 & 1.6029 & 1.7733 & 0.5477 & 0.2807 \\ -11.3819 & 33.4304 & 38.1903 & 5.0016 & -19.7534 & 0.2807 & -33.5526 \end{bmatrix}$$

with corresponding residual

$$\|R_{26}\| = \|\tilde{F} - \tilde{G}_{26}\tilde{X}\tilde{\Lambda} - \tilde{K}_{26}\tilde{X}\| = 5.1978e-006.$$

Therefore, by (14), the optimal approximation solution of Problem II is

$$\hat{G} = \begin{bmatrix} 0.0000 & 13.9810 & -9.2773 & 13.4991 & 0.4572 & -0.1521 & 0.6171 \\ -13.9810 & 0.0000 & -15.4191 & 13.6192 & 1.1087 & -0.5532 & 1.9058 \\ 9.2773 & 15.4191 & 0.0000 & -0.9276 & -8.4668 & 13.5215 & 0.7779 \\ -13.4991 & -13.6192 & 0.9276 & -0.0000 & -13.8359 & 14.0244 & -0.2554 \\ -0.4572 & -1.1087 & 8.4668 & 13.8359 & -0.0000 & -0.3502 & -9.0873 \\ 0.1521 & 0.5532 & -13.5215 & -14.0244 & 0.3502 & -0.0000 & -13.8463 \\ -0.6171 & -1.9058 & -0.7779 & 0.2554 & 9.0873 & 13.8463 & -0.0000 \end{bmatrix},$$

$$\hat{K} = 1000 \times \begin{bmatrix} 1.2061 & 1.8296 & -1.1975 & 1.7926 & -0.0193 & -0.0009 & -0.0114 \\ 1.8296 & 3.6854 & -1.8375 & 1.7684 & -0.0092 & -0.0085 & 0.0334 \\ -1.1975 & -1.8375 & 2.3746 & 0.0020 & -1.1641 & 1.7991 & 0.0382 \\ 1.7926 & 1.7684 & 0.0020 & 7.2087 & -1.7846 & 1.8016 & 0.0050 \\ -0.0193 & -0.0092 & -1.1641 & -1.7846 & 2.4056 & 0.0018 & -1.2198 \\ -0.0009 & -0.0085 & 1.7991 & 1.8016 & 0.0018 & 7.2005 & -1.7997 \\ -0.0114 & 0.0334 & 0.0382 & 0.0050 & -1.2198 & -1.7997 & 2.3664 \end{bmatrix}.$$

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