

# Application of Mapping and Superimposing Rule for Solution of Parabolic PDE in Porous Medium under Cyclic Loading

Mohammad M. Toufigh, and Ahad Ouria

**Abstract**—This paper presents an analytical method to solve governing consolidation parabolic partial differential equation (PDE) for inelastic porous Medium (soil) with consideration of variation of equation coefficient under cyclic loading. Since under cyclic loads, soil skeleton parameters change, this would introduce variable coefficient of parabolic PDE. Classical theory would not rationalize consolidation phenomenon in such condition. In this research, a method based on time space mapping to a virtual time space along with superimposing rule is employed to solve consolidation of inelastic soils in cyclic condition. Changes of consolidation coefficient applied in solution by modification of loading and unloading duration by introducing virtual time. Mapping function is calculated based on consolidation partial differential equation results. Based on superimposing rule a set of continuous static loads in specified times used instead of cyclic load. A set of laboratory consolidation tests under cyclic load along with numerical calculations were performed in order to verify the presented method. Numerical solution and laboratory tests results showed accuracy of presented method.

**Keywords**—Mapping, Consolidation, Inelastic porous medium, Cyclic loading, Superimposing rule.

## I. INTRODUCTION

IN civil engineering projects, construction on saturated incompressible soft soils such as clay and silts is unavoidable. Since the settlement of saturated fine grain soils is a time dependent phenomena, changes of soil properties and time dependent loading such as cyclic loading during of consolidation process would affect the rate of settlement and amount of settlement at any particular time.

Rectangular cyclic loading can be produced by structures such as oil tanks, silos, and reservoirs. If these structures build on saturated clay, the cyclic condition will affect the consolidation process. In the case of normally consolidated (NC) inelastic clays under cyclic loading, consolidation computation will be more complicated, and there would be no exact solution for such condition. Therefore, the solution for consolidation partial differential equation will be more difficult and can be solve by use of numerical techniques.

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There are several methods presented by other researchers for over consolidated (OC) elastic clays under cyclic loading [1, 2, 3, 4, 5, 6, 7]. A semi analytical method presented by M. M. Baligh and Levadux suggests a simple method for such problems only for inelastic clays that is useful just for long period cyclic loads [2].

In this research, a method compound of previous methods is presented to investigate the consolidation of inelastic clays under rectangular cyclic loading. Presented method employs superimposing method along with time variable exchange to solve consolidation of inelastic clays under cyclic loading. Presented analytical solution is suitable for consolidation of both elastic and inelastic clay layers under rectangular cyclic loading.

In the consolidation process, the effect of soil properties change from NC to OC states and reverse conditions during loading and unloading is applied in the calculation by exchange of time variable. Here, inelastic soil body assumed like an elastic one and in order to apply the effect of inelasticity behavior of soil in the solution, the virtual time is used and the calculated results would be related to actual real time. A set of continuous static loads is used instead of time dependant cyclic load according to superimposing rule.

## II. BEHAVIOR OF INELASTIC CLAY UNDER CYCLIC LOADING

Inelastic behavior of soils under a cycle of load can be considered using bilinear model that has shown in Fig. 1.

Since the coefficient of compressibility and permeability of inelastic clay changes during a loading and unloading condition, the coefficient of consolidation ( $c_v$ ) is a function of these parameters and would change in each cycle of loading. By assumption that  $c_v$  has a constant value during state of NC or state of OC and changes suddenly when the soil body transmits from NC to OC or reverse condition.

At first half cycle of loading, soil is at NC condition and stress path is according to (1-2) route, as shown in the Fig. 1. During all of unloading half cycles, soil is at OC condition and stress path is according to (3-4) route.

After the first cycle, in the following half cycles, stress path will be according to (4-5-6) route. Position of node 5 is equal to effect of precompression pressure ( $P_c$ ), which imposed by previous cycle to the soil. This  $P_c$  would increases by number of cycles and reaches to a position where all of the soil in the

loading half cycle stay at OC condition, which would cite as steady-state condition.

Since  $c_v$  changes at point 5 and 3 according to bilinear model in Fig. 1, one of the main objectives of this problem is to determine the required time in order to get to point 5 in every half cycle of loadings. Point 3 is at beginning of unloading half cycle.

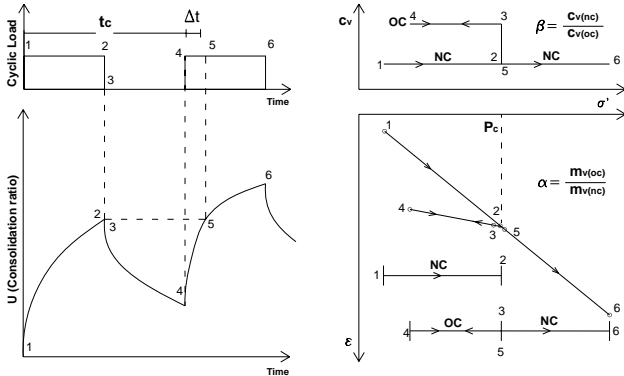


Fig. 1 Inelastic behavior of clay under cyclic load

### III. DETERMINATION OF TIME SPACE MAPPING FUNCTIONS AND P.D.E SOLUTION

Based on compatibility of fluid outflow and volume change of soil body, governing partial differential equation for one-dimensional consolidation phenomena under a time dependent loading is:

$$c_v \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} + \frac{\partial p}{\partial t} \quad (1)$$

The term  $\partial p / \partial t$  is the effect of the load variation with the time and  $c_v$  is consolidation coefficient.

In this research, the effect of load variation with time is adopted by superimposing method, which will introduce in next section and  $c_v$  variation is applied in solution by time modification. Therefore, Terzaghi's consolidation theory is used for calculations.

Analytical solutions of one-dimensional consolidation partial differential equation with uniform initial pore water pressure distribution based on Terzaghi's consolidation theory are according to Equation 2 and 3 for pore water pressure distribution and degree of consolidation respectively.

$$u(t) = \sum_{m=0}^{\infty} \frac{2u_0}{M} \exp(-M^2 T) \sin\left(\frac{M \cdot z}{H}\right) \quad (2)$$

$$M = \frac{(2m+1)\pi}{2}, \quad T = \frac{c_v \cdot t}{H_d^2}$$

$$U(t) = 1 - \sum_{m=0}^{\infty} \frac{2}{M^2} \exp(-M^2 T) \quad (3)$$

Where,  $u$  and  $U$  are pore water pressure and degree of

consolidation respectively.

Since time factor ( $T$ ) is a function of  $c_v$  and  $t$ , it means that equal variation of both factors would have same changes on results. Therefore, according to Equation 4, for equal results any changes for value of  $c_v$  require similar adjustment for value of  $t$ .

$$T_v = \frac{(kc_v)t}{H_d^2} = \frac{c_v(kt)}{H_d^2} = \frac{c_v t'}{H_d^2}, \quad t' = kt \quad (4)$$

Where:  $t$  is real time variable,  $t'$ , is virtual time variable and,  $k$ , can be any factor.

Therefore, for consolidation calculations of a clay layer with variable  $c_v$ , it can be substituted with constant  $c_v$  and adjusted time space which would introduces a mapping function for time space.

During the period of the unloading half cycles where the soil is at OC state, the value of  $c_v$  is different than its value at NC state. Calculations would be done during unloading period with NC value of  $c_v$  by time adjustment, in order to influence the effect of  $c_v$  variation. Therefore equivalent time for unloading half cycles would be:

$$t'_N = \frac{t_c}{2\beta}, \quad N = 2, 4, 6, \dots \quad (5)$$

Where,  $t_c$  and  $\beta$ , are introduced in Fig. 1 and  $N$  is number of half cycle.

Equation 5 introduces mapping function for time variable for unloading half cycles.

After the first full cycle (a pair of loading and unloading half cycles), in the following half cycles of loading, at beginning, the soil is at OC state until the average degree of consolidation reaches to previous maximum degree of consolidation, which is equal to its value at end of last half cycle of loading.

In Fig. 2,  $\Delta t_N$  is the time portion of each loading half cycle that soil is at OC state (according to route (4-5) in Fig. 1) and afterward becomes NC.

By adoption of superimposing rule, consolidation of inelastic soil layer under cyclic loading would be analyzed same as elastic one by imposing computed virtual time

Fig. 2 shows a rectangular cyclic loading system which is adapted in virtual time space by a set of static loads based on superimposing rule.

Equation 6 can be used to calculate  $\Delta t'_N$ , using Fig. 2, where  $\Delta t'_N$  is virtual time in mapped space, which is related to  $\Delta t_N$  ( $\Delta t'_N = \Delta t_N / \beta$ ). At the end of  $\Delta t_N$ , (according to point 5 in Fig. 1 in each loading half cycle excepting first one) the soil status changes from OC to NC condition. The average degree of consolidation at end of  $\Delta t_N$  must be equal to its previous maximum value.

$$\sum_{n=1}^{N-1} (-1)^{n+1} U(t'_{i,n-1} + \Delta t'_N) = U_{c_{N-2}}, \quad t'_{i,n-1} = \sum_{i=n}^{N-1} t'_i \quad (6)$$

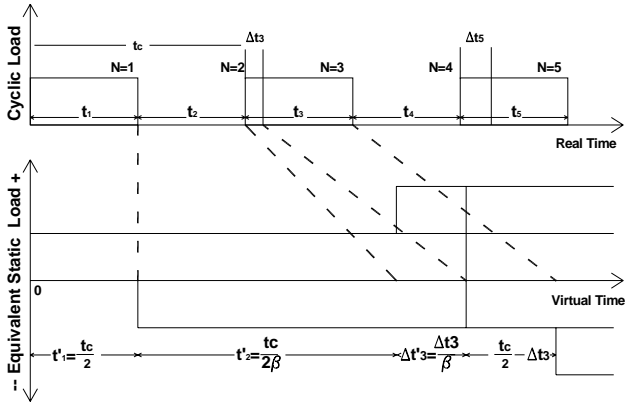


Fig. 2 Cyclic loading adapted by superimposing rule mapped on virtual time space

The left side of Equation 6 is actually degree of consolidation at the end of  $\Delta t'_N$  which would compute based on superimposing method. The right side of Equation 6 is maximum degree of consolidation at previous cycle.

At first stage for  $N=3$ , in the above equation only  $\Delta t'_3$  is unknown, which can be calculated.  $U$ , would be calculated by equation 3.  $U_c$  is degree of consolidation under cyclic load, where in first half cycle of loading, it is equal to degree of consolidation under static load for  $t = t_1$ .

When  $\Delta t'_3$  calculated, virtual time for second half cycle of loading would be calculated for  $N=3$  in the following equation.

$$t'_N = \Delta t'_N + \frac{t_c}{2} - \beta \Delta t'_N, \quad N = 3, 5, 7, \dots \quad (7)$$

$\Delta t'_N$  is the time that the  $c_v$  changes from OC to NC state in each loading half cycle. Therefore, only one of two time portions of loading half cycle (before or after of  $\Delta t'_N$ ) must be adopted depending on usage of NC or OC value of  $c_v$  in calculations. In Equation 7, in order to calculate the virtual time for loading half cycle, first portion of time is adopted. The equation 7 is mapping function in time space for loading half cycles.

When virtual time for second loading half cycle,  $t'_3$ , calculated, pore water pressure ( $u_c$ ) and degree of consolidation ( $U_c$ ) at end of second half cycle of loading would be determined by Equations 8 and 9 which cyclic degree of consolidation is required for calculation of  $\Delta t'_5$  in the third half cycle of loading, and so on.

$$u_{CN} = (-1)^N \sum_{n=1}^N (-1)^n u_{(t'_{nN})} \quad t'_{nN} = \sum_{i=n}^N t'_i \quad (8)$$

$$U_{CN} = (-1)^N \sum_{n=1}^N (-1)^n U_{(t'_{nN})} \quad (9)$$

Above procedure would be repeated for determination of the  $\Delta t'_N$ ,  $t'_N$ ,  $u_c$  and  $U_c$  for the following cycles.

The above procedure can be summarized by a step-by-step systematic method that is as follow:

1. Cyclic consolidation degree at end of first half cycle of loading should be calculated using Equation 9. This would always be equal to  $U$  (Equation 3) under static loading for period of  $t_1$ .
2. Virtual times for unloading half cycles that have a constant value would calculate from Equation 5.
3. By using Equations 6 and 7,  $\Delta t'_3$  and  $t'_3$  would be calculated respectively.
4. Pore water pressure and degree of consolidation at end of second half cycle of loading would be determined by using Equations 8 and 9.

Steps 2 to 4 should be repeated for next cycles.

Settlement at end of loading and unloading half cycles would be calculating using Equations 10 and 11 respectively.

$$S_N = U_{cN} \cdot m_v \cdot p \cdot H, \quad N = 1, 3, 5, \dots \quad (10)$$

$$S_N = S_{N-1} + (U_{cN} - U_{cN-1}) \cdot m_v \cdot \alpha \cdot p \cdot H, \quad N = 2, 4, 6, \dots \quad (11)$$

In the above equations,  $S_N$  is settlement at end of loading or unloading half cycle.  $U_{cN}$  is degree of consolidation under cyclic loading obtained from Equation 9,  $m_v$  is compressibility coefficient in NC condition,  $p$  is cyclic load intensity,  $H$  is clay stratum thickness and  $\alpha$  is the ratio of compressibility coefficient at OC condition to its value at NC condition.

#### IV. IMPLEMENTATION OF PRESENTED METHOD

Following example describes the procedure of calculations using presented method.

Consolidation of an inelastic normally consolidated clay layer with  $c_v = 0.0029 \text{ cm}^2/\text{min}$ ,  $H=2.826\text{cm}$ , with  $\beta = 0.95$ , full cycle period of  $t_c = 30\text{min}$  is desired for calculation by presented method. Calculation procedure would be as follow:

At first stage, cyclic degree of consolidation at end of first half cycle of loading ( $U_{c1}$ ) will be calculated.

As declared before, first half cycle of loading is same as a static load system therefore, using Equation 3 for  $t=15\text{min}$  or  $T_1 = 0.0218$ , where  $t$  and  $T$  are time and time factor at end of first half cycle,  $U_{c1}$  would be equal to 0.166.

In first half cycle, because the soil body is NC during all of loading time, the value of  $c_v$  would be constant. So virtual and real times are equal to:

$$T_1 = T'_1 = 0.0218$$

Equivalent virtual times for unloading half cycles ( $N=2, 4, 6 \dots$ ) have a constant value that would be calculated by Equation 5:

$$T'_N = \frac{T_c}{2\beta} = \frac{T_1}{\beta} = \frac{0.0218}{0.95} = 0.229, \quad N = 2, 4, 6, 8, \dots$$

Cyclic degree of consolidation at end of second half cycle (end of first full cycle) would be calculating based on

superimposing rule, using Equation 9.

Cyclic load during of first cycle would be replaced by two static loads that effect by  $(T_1' + T_2')$  duration with positive sign and  $T_2'$  with negative sign. Therefore, degree of consolidation at end of second half cycle is equal to:

$$U_{C2} = U(T_1' + T_2') - U(T_2') = U(0.0218 + 0.2294) - U(0.2294) = 0.5635 - 0.5392 = 0.0243$$

For calculation the cyclic degree of consolidation at end of third half cycle,  $\Delta T_3'$  is required, which would be calculated by using Equation 6.

As mentioned before, cyclic degree of consolidation at end of  $\Delta T_3'$  from beginning of third half cycle must be equal to  $U_{C1}$ .

Therefore, using equation 6 can write:

$$U(T_1' + T_2' + \Delta T_3') - U(T_2' + \Delta T_3') + U(\Delta T_3') = U_{C1}$$

$$U(0.0218 + 0.2293 + \Delta T_3') - U(0.2293 + \Delta T_3') + U(\Delta T_3') = 0.1666$$

Therefore:

$$\Delta T_3' = 0.0161$$

$T_3'$ , would be calculate using Equation 7:

$$T_3' = \Delta T_3' + \frac{T_c}{2} - \beta \Delta T_3' =$$

$$0.0161 + 0.0218 - 0.095 \times 0.0161 = 0.0364$$

Degree of consolidation under cyclic load would be calculated by substituting known values of virtual times in Equation 9 which will be as follow:

$$T_{13}' = T_1' + T_2' + T_3' = 0.0218 + 0.2293 + 0.0364 = 0.2875$$

$$T_{23}' = T_2' + T_3' = 0.2293 + 0.0364 = 0.2657$$

$$T_{33}' = T_3' = 0.0364$$

$$U_{C3} = U(0.2875) - U(0.2657) + U(0.0364) = 0.2377$$

Above procedure can be repeated to compute the virtual times and consolidation degree for other cycles. Following table shows results of calculations.

TABLE I  
SUMMARY OF NUMERICAL CALCULATIONS BY PRESENTED METHOD

N	$\Delta T'_N$	$T'_N$	$U_{c_N}$
1	0	0.02179	0.166565
2	0	0.229368	0.024304
3	0.016291	0.036533	0.237785
4	0	0.229368	0.052483
5	0.028327	0.047426	0.292253
6	0	0.229368	0.077542
7	0.038993	0.057078	0.336726
8	0	0.229368	0.098929
9	0.049062	0.066191	0.374081
10	0	0.229368	0.117297
11	0.058678	0.074894	0.406033
52	0	0.229368	0.260346
53	0.177107	0.182072	0.645401
100	0	0.229368	0.287748
101	0.219156	0.220126	0.695289
200	0	0.229368	0.293916
201	0.229019	229052	0.705528

## V. PRESENTED METHOD VERIFICATION

### A. By Finite Difference Method

In order to verify the presented analytical method, it should be compared with other procedures. At first stage in this research finite difference numerical method (based on Crank-Nicholson method) was adopted for consolidation of a clay layer under a cyclic loading with time factor of  $T_c=0.01$  and  $\beta=0.1$ .

Results of presented analytical and numerical method are illustrated in Fig. 3. It would be seen that presented analytical method and numerical method results are coincide together where discrimination of the result in Fig. 3 is impossible.

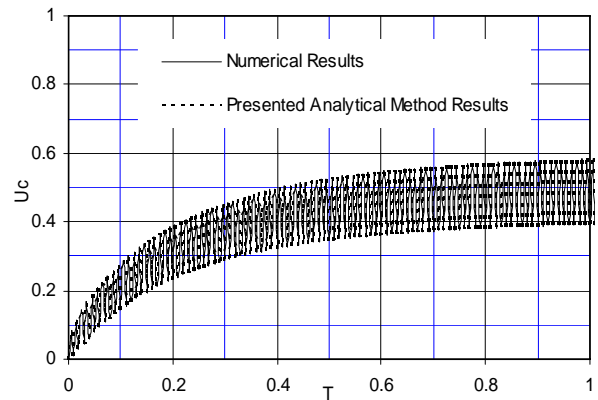


Fig. 3 Verification presented method by finite difference method.

In Fig. 4, difference between analytical and numerical method is illustrated. It can be seen that the amount of differences is very small and it is negligible. Differences between presented analytical method and numerical results is less than 0.0025 at beginning of solution and less than 0.0005 in steady-state condition. The amount of error is very small

and it is due to finite difference mesh coarsens.

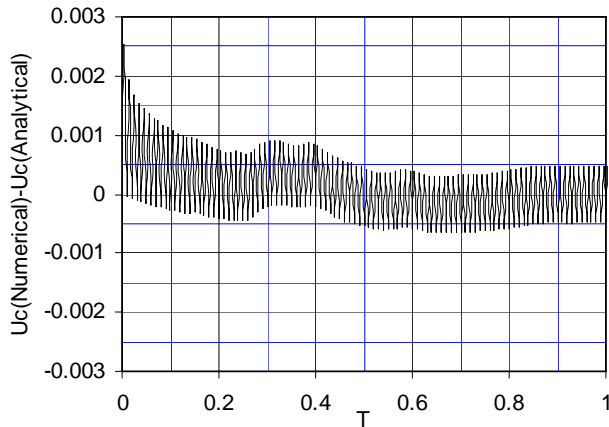


Fig. 4 Differences between presented method and numerical method results

**B. By Laboratory Results**

A series of laboratory tests have performed to investigate the consolidation of inelastic clays under cyclic loading. Tests were performed in one-dimensional consolidation apparatus (Toufigh and Ouria [10, 11]). Clay samples were taken from Nouq field at Kerman province in Iran. These samples were taken from depths up to 300m.

Specifications of 20 specimens are shown in Table II.

TABLE II  
SPECIFICATIONS AND TEST CONDITIONS FOR LABORATORY TESTS

Test No	P kPa	Tc	H <sub>0</sub> cm	m <sub>v</sub> cm <sup>2</sup> /kN	c <sub>v</sub> cm <sup>2</sup> /min	β	α
1	25	60	1.38	1.271	0.0012	0.095	0.09
2	100	60	1.38	1.176	0.0021	0.1	0.11
3	50	20	1.37	2.041	0.0005	0.07	0.1
4	50	30	1.43	3.356	0.0014	0.1	0.1
5	50	60	1.42	3.532	0.0014	0.1	0.1
6	75	20	1.38	2.823	0.0014	0.105	0.11
7	75	40	1.38	2.705	0.0014	0.105	0.11
8	50	60	1.37	2.487	0.0008	0.1	0.1
9	100	60	2.7	0.808	0.0015	0.09	0.09
10	100	60	2.84	0.792	0.0020	0.093	0.09
11	60	30	2.59	0.595	0.0011	0.065	0.11
12	60	60	2.64	0.601	0.0011	0.065	0.11
13	62.5	20	2.78	0.777	0.0015	0.08	0.1
14	62.5	40	2.77	0.763	0.0015	0.08	0.1
15	50	30	2.83	0.750	0.0029	0.095	0.095
16	50	60	2.84	0.775	0.0029	0.095	0.095
17	200	30	2.9	0.259	0.0096	0.09	0.09
18	200	20	2.92	0.257	0.0096	0.09	0.09
19	100	30	2.88	0.208	0.00261	0.09	0.09
20	100	60	2.90	0.207	0.00261	0.09	0.09

Where: Tc = time factor for a cycle, H<sub>0</sub> = initial height, P=cyclic load, m<sub>v</sub>=compressibility coefficient, c<sub>v</sub>=consolidation coefficient, β=cv(nc)/cv(oc), α=mv(nc)/mv(oc)

The ratio of  $T_c/\beta$  is a factor for estimating the cyclic load frequency [2].

For  $T_c/\beta$  of 1 and more means relatively longer period of cyclic load and for 0.02 and less means relatively shorter period of cyclic load. In these tests, dimensions of standard consolidation ring cell have modified to obtain relatively rapid cyclic loading condition. The height of consolidation cells was increased by two times of original size.

Fig. 5 shows a laboratory cyclic consolidation test result along with presented analytical solutions. These results are based on sample number 13 from Table I with double drained condition.

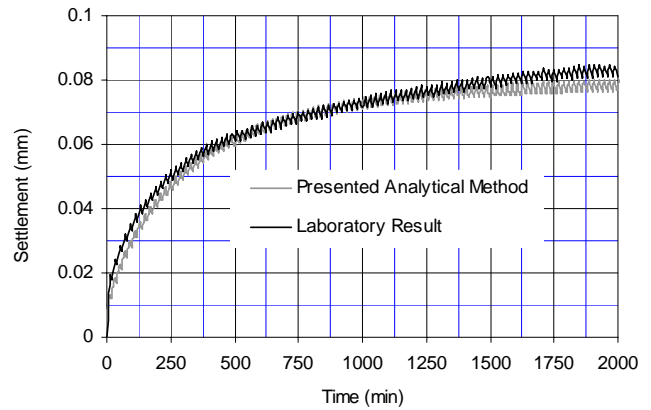


Fig. 5 Verification presented method by laboratory results (test no. 13)

Test was performed with,  $T_c/\beta$  of 0.2 which is relatively at rapid condition or relatively high frequency. As shown in Fig. 5, presented analytical method results are close to laboratory results. The difference between these two is mainly due to laboratory errors.

For another alternative, another double drained system test result is illustrated in Fig. 6 for sample number 1.

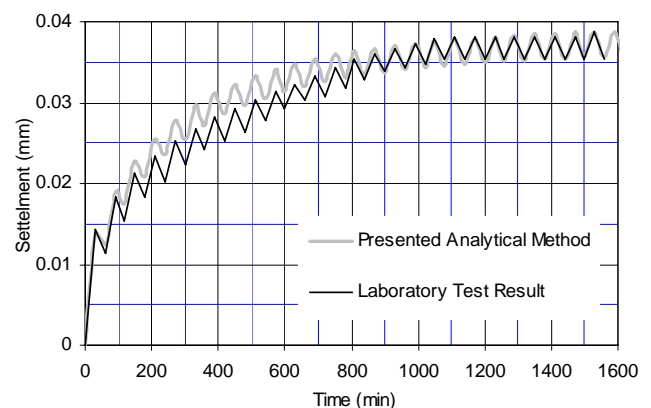


Fig. 6 Verification presented method by laboratory results (test no. 1)

It can be seen from Fig. 6 that presented method results are close to laboratory results same as Fig. 5. It should be noted that  $T_c/\beta$  for this test has higher value comparing to sample

no13. Above two tests was in different cyclic period. In the issue of cyclic consolidation of inelastic clays, loading period is the most important parameter that affects the behavior of procedure. Above tests are partially in different conditions. It would be seen that the result of presented method is independent to loading frequency.

Results of laboratory tests are abbreviated in Fig. 7. In this figure, results of presented method are illustrated along with laboratory results. For easy comparing, just the final values of consolidation degree obtained from laboratory tests and presented method are plotted. Horizontal axis is dimensionless cyclic load time factor which is representation of loading condition and soil material.

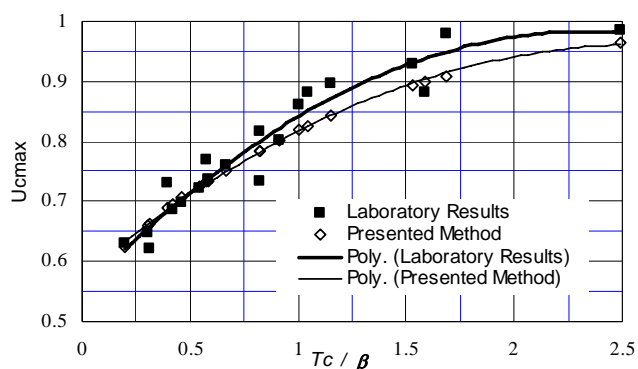


Fig. 7 Summary of laboratory and presented method results

It can be seen that for all values of cyclic load time factor, results of presented method are in a good correlation with laboratory results. It would be conclude that the presented method would be an accurate solution for consolidation of inelastic soils under cyclic load.

Since in presented method, mapping functions are function of material properties, therefore in the case of elastic materials, it is enough to substitute elastic properties ( $\beta=1$ ) in presented relationships. As another privileges of presented method, it would be used for both elastic and inelastic materials.

## VI. CONCLUSION

In this paper, consolidation of inelastic clays under cyclic loads is investigated. A new method based on superimposing rule on virtual time space presented. Previously superimposing rule had used for consolidation of elastic materials under cyclic loading. Since material property changes during cyclic load, the time variable exchanged by virtual time variable. In the virtual time space, the effect of the change of material properties applied to solution by adopting a new time and duration, which the loads affect on soil, therefore this would introduce an elastic material. By use of virtual time, superimposing method became applicable for inelastic problems. Calculations have done in virtual time and results related to real time. Numerical method used to verification of presented method. Presented method results are same as numerical method results. More than 20 laboratory

tests performed for verification of presented methods. Calculations based on presented analytical method have a good correlation with laboratory results. An exciting aspect of presented method is its reliability for all loading periods and material properties.

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