Asymptotic Stability of Input-saturated System with Linear-growth-bound Disturbances via Variable Structure Control : An LMI Approach

Yun Jong Choi, Nam Woong Kong and PooGyeon Park

***Abstract*—** Variable structure control (VSC) is one of the most useful tools handling the practical system with uncertainties and disturbances. Up to now, unfortunately, not enough studies on the input-saturated system with linear-growth-bound disturbances via VSC have been presented. Therefore, this paper proposes an asymptotic stability condition for the system via VSC. The designed VSC controller consists of two control parts. The linear control part plays a role in stabilizing the system, and simultaneously, the nonlinear control part in rejecting the linear-growth-bound disturbances perfectly. All conditions derived in this paper are expressed with linear matrices inequalities (LMIs), which can be easily solved with an LMI toolbox in MATLAB.

***Keywords*—** Input saturation, linear-growth bounded disturbances, linear matrix inequality (LMI), variable structure control

I. Introduction

SINCE 1970's, VSC has gained much attention for being one of the useful design tools to control various practical systems with uncertainties and disturbances [1]-[4]: nonlinear system, MIMO systems and even stochastic systems.

In the engineering systems, difficulties exist in considering input saturation and disturbances simultaneously. The previous researches [5]-[7] have tried to handle these difficulties using linear-feedback techniques; however, there was still a defect so that they did not perfectly reject the disturbances but attenuated them. The best approach to complement the defect is to use VSC that is robust or insensitive against the disturbances, but there has been no academic research handling both the disturbances and input saturation, simultaneously, via VSC. In addition, in the aspect of the type of disturbances, $_2$- or $_\infty$-type disturbances in the system are usually considered, but the linear-growth-bound disturbance, *i.e* $\| \; (\; , \;)\| \leq \; \| \; \| + \;$, which is bounded to both external noises and systemical states might not often be regarded. It is notable that $(\; , \;)$ is linearly bounded by the norm of the states and it gives rise to the internal and external effects to the system. Although [8]-[9] considered the linear-growth-bound disturbances by VSC, they did not handle the input-saturated system in the literature.

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Therefore, this paper proposes a VSC for input-saturated systems with linear-growth-bound disturbances to achieve the asymptotic stability, in the sense of Lyapunov stability, whose controller consists of two control parts: *linear control part* stabilizing the close-loop saturated system and *nonlinear control part* perfectly rejecting the linear-growth-bound disturbances. All the conditions for designing the controller are expressed with Linear Matrices Inequalities (LMIs) so that all solutions are easily calculated with an LMI toolbox in MATLAB.

This paper is organized as follows. Section II will address the problem statement. Section III will explain the proposed method which achieves asymptotic stabilization for the given system against linear-growth-bound disturbances via VSC. Section IV will show the performance of the resulting controllers in the literature through an example. Finally, section V will conclude with a summarization and future work.

II. Problem Statement

Consider the input-saturated system

$$\dot{\;}(\;) \;=\; (\;) + \; \mathrm{sat}(\; (\;)) + \; (\;), \qquad (1)$$

where $(\;) \in \mathbf{R}^n$, $(\;) \in \mathbf{R}^m$, and $(\;) \in \mathbf{R}^m$. The disturbance term $(\;)$ satisfies both the so-called matching condition and the linear growth bound such as

$$(\;) = \; (\; , \;), \quad \| \; (\; , \;)\| \leq \; \| \; \| + \; , \qquad (2)$$

where $\;0$ is a known constant and $\;0$ is a known scalar-valued function. And sat( ) denotes a saturation operator with level yielding

$$[\mathrm{sat}\;(\;)] \triangleq \begin{cases} & \text{if } \; | \; | \leq \\ & \text{if} \\ & \text{if} \end{cases} \qquad (3)$$

In this case, one can find the following relation [6]:

$$\mathrm{sat}\;(\;) \;\in\; \mathbf{Co}\{ \;\; _j \;+\; \bar{\;}_j \;\; | \;\; \in [1, 2^m] \;\; , \qquad (4)$$

where is a vector yielding $| \; _j| \leq \;$, $_j$ denotes a diagonal matrix with all possible combinations of 1 and 0 diagonal entries, $\bar{\;}_j \triangleq \;\; _j$ and **Co** is the convex hull, which is defined as follows.

Proposition 2.1 (Convex hull): The convex hull of a set is the minimal convex set that contains . For a group of points $_1, \; _2, \; \; , \; _n \in \; ^n$, the convex hull of these points is described as

$$\mathbf{Co}\{ \; _1, \; _2, \; \; , \; _n \;= \left\{ \sum_{i=1}^{n} \alpha_i \; _i : \sum_{i=1}^{n} \alpha_i = 1, \alpha_i \;\; 0 \right\} \qquad (5)$$

III. Main Result

To stabilize the system (1), we shall propose a new VSC method, which is described as the following theorem with the LMI terms.

Theorem 3.1: The state $(\ )$ in $\mathcal{E} \triangleq \{ \ | \ ^T \ \ 1\}$ asymptotically converges to the origin if there exist $\bar{\ }$, , $\bar{\ }$ and $\bar{\ }$ 0 such that for all $\in [1, 2^m]$ and $\in [1, \ ]$

$$\begin{bmatrix} (1,1) & \bar{\ } \\ & \bar{\ } \end{bmatrix} \quad 0, \qquad (6)$$

$$(1,1) = \bar{\ } + \ _j \bar{\ } + \ \bar{_j} \bar{\ } + \bar{\ } \ ^T + \bar{\ }^T \ ^T_j \ ^T + \bar{\ }^T \ \bar{_j}^T \ ^T,$$

$$\begin{bmatrix} \Omega & \bar{\ } \ \frac{b^2\bar{\zeta}}{2} \ ^T \\ \bar{\ }^T \ \frac{b^2\bar{\zeta}}{2} & \bar{\ } \end{bmatrix} \quad 0, \qquad (7)$$

where $\Omega_{ii} \leq (\quad)^2$, $\bar{\ } \triangleq \ ^{-1}$, $\bar{\ } \triangleq \ ^{-1}$, $\bar{\ } \triangleq \ ^{-1}$ and $\bar{\ } \triangleq \ ^{-1}$. In this case, the controller is

$$(\ ) = \bar{\ } \bar{\ }^{-1} (\ ) + \bar{\ }(\ ), \qquad (8)$$

where the $\bar{\ }(\ )$ is defined as

$$\bar{\ }(\ ) \triangleq \frac{2}{2} (\ ,\ ) \frac{}{\|\ (\ ,\ )\|} (\ ,\ ), \qquad (9)$$

$$(\ ,\ ) \triangleq \ ^T \ (\ ) \qquad (10)$$

Proof: To stabilize (1) by VSC, let us consider a Lyapunov function $(\ (\ )) = \ ^T(\ ) \ (\ )$ and the derivative of $(\ (\ ))$ is

$$\dot{\ }(\ (\ )) = 2 \ ^T(\ ) \ \{ \ (\ ) + \ \{\text{sat}(\ (\ )) + (\ ,\ ) \qquad (11)$$

In this case, we shall consider $\mathcal{E}$, which is the so-called *attraction region* by properly designing the input $(\ )$. If we design the input $(\ )$ and the auxiliary input $(\ )$ such as

$$(\ ) = \ + \bar{\ }, \ (\ ) = \ + \bar{\ }, \qquad (12)$$

where the linear control part stabilizes the system, then the nonlinear control part $\bar{\ }$ rejects the disturbance, and is the auxiliary feedback matrix. Then (4) is rewritten as

$$\text{sat}(\ (\ ) + \bar{\ }(\ )) \quad \sum_{j=1}^{2^m} \alpha_j \{ \ _j \ + \ \bar{_j} \ \} (\ ) + \bar{\ }(\ ) \qquad (13)$$

Thus, for all $\in [1, 2^m]$, (11) can be described as

$$\dot{\ }(\ (\ )) = \ ^T(\ ) \sum_{j=1}^{2^m} \alpha_j \{ \ (\ + \ _j \ + \ \bar{_j} \ ) + (\ + \ _j \ + \ \bar{_j} \ )^T \ (\ ) + 2 \ ^T(\ ,\ ) \{\bar{\ }(\ ) + (\ ,\ ) \ ,$$

where should be chosen as

$$| \ (\ ) + \bar{\ } | \leq \qquad (14)$$

From [9], using (2) and (9), we can achieve

$$2 \ ^T(\ ,\ ) \{\bar{\ }(\ ) + (\ ,\ ) \ \leq \ \| \ \|^2$$

Therefore, we obtain the resulting condition such that

$$\dot{\ }(\ ) \leq \ ^T(\ ) \sum_{j=1}^{2^m} \alpha_j \{ \ (\ + \ _j \ + \ \bar{_j} \ ) + (\ + \ _j \ + \ \bar{_j} \ )^T \ (\ ) + \| \ \|^2 \qquad (15)$$

And to ensure (14), the following inequality needs to hold for all $(\ ) \in \mathcal{E}$:

$$\left| \ \frac{2}{2} \ ^T \ \right| \leq \ , \qquad (16)$$

which is ensured if it holds

$$\begin{bmatrix} \Omega & \frac{b^2}{2\zeta} \ ^T \\ ^T \ \frac{b^2}{2\zeta} & \end{bmatrix} \quad 0 \qquad (17)$$

Then (6) and (7) can be obtained with some efforts. ∎

IV. Numerical Example

In this section, we will show the the performance of the proposed controller through an example. The system parameters for this simulation are as follows:

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \quad = 1, \quad = 0\,1, \quad = 0\,1$$

Fig. 1 illustrates both the attraction region and state transition when the proposed controller is applied, where the initial state locates at (-6,4). As shown in the figures, the initial state successfully goes toward the origin without departing from the attraction region in the presence of the linear-growth-bound disturbances $(\ )$.

Fig. 2 depicts the trajectories of the states and input under the same example as time passes. Like Fig. 1, all states asymptotically converge to the origin, where the input is saturated at $= 1$ and then is fast switching so as to perfectly reject the disturbances.

Fig. 3 shows the size of the attraction region according to the increasing , which is the coefficient of the first degree of linear-growth-bound disturbances $(\ )$. Through the figure, it is expected that the more the disturbance bound increases, the less the attraction region is obtained.

V. Conclusion

In this paper, we proposed a new VSC method for input-saturated systems with linear-growth-bound disturbances. The VSC is a very useful tool for dealing with uncertainties and disturbances; however, up to now, not enough studies on the input-saturated system with linear-growth-bound disturbances via VSC have been constructed so far. This paper suggested how VSC was applied in input-saturated system and how the controller was designed. The proposed controller was composed of two control parts. The linear control part played a role in stabilizing the system, and simultaneously, the nonlinear

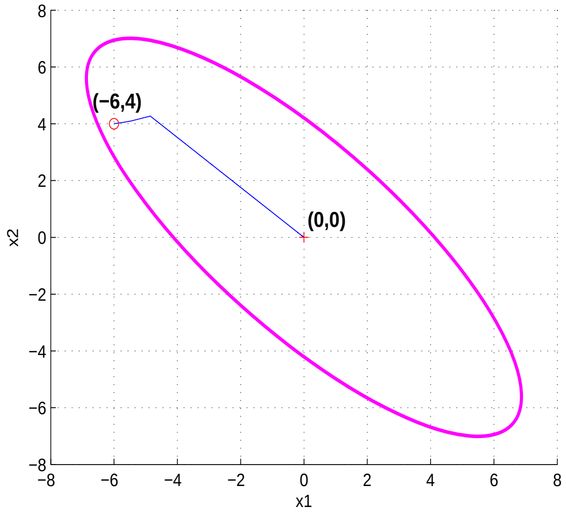

Fig. 1. The attraction region and state transition

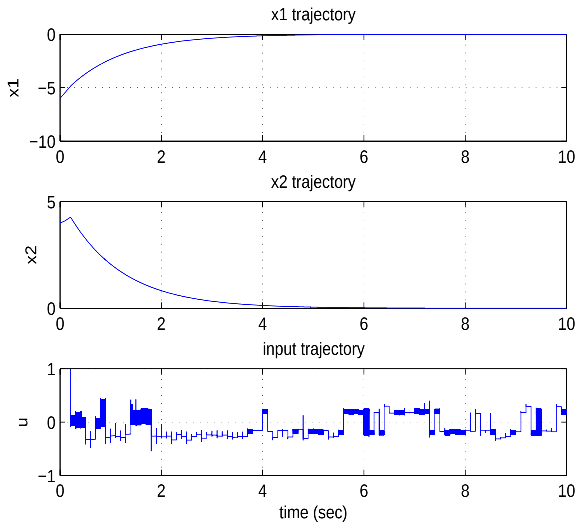

Fig. 2. The states and input trajectories

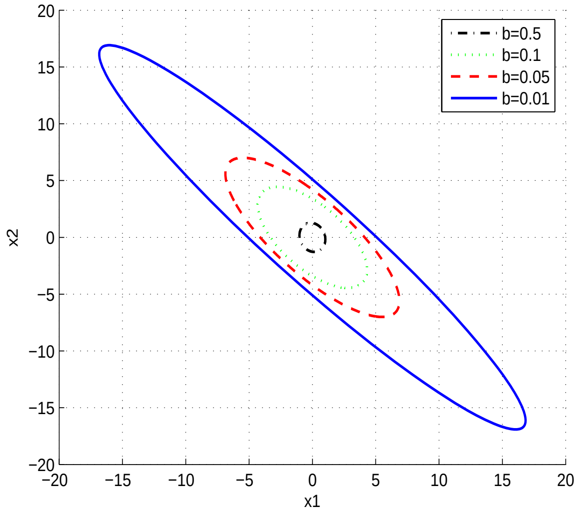

Fig. 3. The size of attraction region according to increasing b

control part in rejecting the linear-growth-bound disturbances perfectly. In the main result, we derived the LMI conditions, which achieve the asymptotic stability via VSC, in which the conditions could be easily solved with the LMI toolbox in MATLAB. The simulation results showed the performance of the proposed controller via VSC. For further work, we will extend the controller design from the state-feedback case to the output feedback one. In that case, it is expected that the proposed method can be applied to various practical systems.

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