

# Block Activity in Metric Neural Networks

Mario Gonzalez, David Dominguez, and Francisco B. Rodriguez

**Abstract**—The model of neural networks on the small-world topology, with metric (local and random connectivity) is investigated. The synaptic weights are random, driving the network towards a chaotic state for the neural activity. An ordered macroscopic neuron state is induced by a bias in the network connections. When the connections are mainly local, the network emulates a block-like structure. It is found that the topology and the bias compete to influence the network to evolve into a global or a block activity ordering, according to the initial conditions.

**Keywords**—Block attractor, random interaction, small world, spin glass.

## I. INTRODUCTION

NEURONS never function in isolation; they are organized into ensembles or neural circuits that process specific kinds of information and provide the foundation of sensation, perception and behavior[1]. Our brain consists of various areas performing special tasks and communicating along specific pathways. Even on a smaller scale it is organized into layers and columns. Furthermore on an even smaller scale, neurons seem to interact in a rather disordered fashion, and the pathways between different areas are to some degree diffuse. The storage of information (firing patterns) or the stimulus-response schemes in neural networks can formally be described as the construction of attractors in the dynamics of spin systems[2]. This construction is achieved by giving suitable values to the exchange couplings between the spins, which take the role of synapses in neural networks.

In this work a model of sparsely connected Hopfield-type neural networks[2] on the small-world topology of Watts-Strogatz[3] is presented. The concept of “small world” networks was introduced as an attempt to capture and study nontrivial features observed in realistic social, biological and technological networks. The key idea is to generate a structure which interpolates between a regular lattice and a random graph. One starts with a locally connected network, e.g. a ring, of nearest neighbours, and subsequently “re-wires” randomly with probability  $\omega$  those local connections, creating long-range shortcuts[4].

The response of a network to a given input stimulus leads to a particular configuration of the neural activity. If there is an excitatory bias in the synaptic weights, the neurons will be ordered in either the active or the inactive states, according to the initial conditions (the stimulus)[5]. If the stimulus is a more complex set of neural activities, the network may become trapped in a stationary state with no global ordering. However, a local order can emerge if the stimulus has

some neighbourhood structure, whatever the network topology preserves some metric, with stronger connectivity between nearest neurons than between neurons far from each other[6]. The block-like attractor structure of the network is explored, to study its dynamical behaviour, and the parameters which allow such configurations.

## II. THE MODEL

In order to describe the neural activity the following model is used. A neuron  $i$  can be in one of two states, firing or nonfiring, described by binary variables  $\sigma_i \in \{\pm 1\}$  (active/inactive). The state of a neuron  $\sigma_i$  is updated in time  $t$  through the following equation:

$$\sigma_i(t) = \text{sign} \left( \sum_i h_i(t-1) - \theta \right), \quad h_i(t) \equiv \sum_j J_{ij} \sigma_j(t) \quad (1)$$

where  $h_i(t)$  is the postsynaptic field arriving at neuron  $\sigma_i$ . Here sign denotes the sign function defined as:  $\text{sign}(z) = 1$  if  $z \geq 0$ , and  $\text{sign}(z) = -1$  if  $z < 0$ . The variable  $\theta$  is the firing threshold which is considered to be zero. A synchronous update and asymmetric weights are used.

The synaptic couplings between neurons  $i, j$  are  $J_{ij} \equiv C_{ij} W_{ij}$ , where  $\mathbf{C} = \{C_{ij}\}$  is the topology matrix and  $\mathbf{W} = \{W_{ij}\}$  are the synaptic weights. The topology matrix, with  $C_{ij} \in \{0, 1\}$  splits in local and random links. The local links connect each neuron to its  $K_L$  nearest neighbours, in a closed ring. The random links connect each neuron to  $K_R$  others uniformly distributed along the network[7]. Hence, the network degree is  $K = K_L + K_R$ . The network topology is then characterized by two parameters: the *connectivity* ratio, and the *randomness* ratio, defined respectively by:

$$\gamma = K/N, \quad \omega = K_R/K, \quad (2)$$

where  $\omega$  plays the role of a rewiring probability in the *small-world* model[3].

The weights  $W_{ij}$ , of the connections between neurons  $i$  and  $j$ , are composed of two terms:

$$W_{ij} = \overline{W} + W_{ij}^r, \quad (3)$$

where  $W_{ij}^r$  are generated randomly to be either +1 or -1 with equal probability, representing either an excitatory or an inhibitory synapse, respectively. One defines a variable  $W \in (0, 1)$ , and adds  $W$  to the weight matrix  $W_{ij}$  in order to induce a bias (ferromagnetic) interaction in the network[8].  $W$  is the same for all synapses.

One wants to study the evolution of the network when initialized in blocks ( $l_+$  and  $l_-$ ). The blocks are defined as the groups of neighbour neurons initialized as  $\sigma_i = +1, i \in l_+$  and  $\sigma_i = -1, i \in l_-$ . A mesoscopic variable  $A_l(t)$  is used to

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describe the neural activity of block  $l$ , with size  $N_l$ , as the fraction of neurons firing at time  $t$ ,

$$A_l(t) = \frac{1}{N_l} \sum_{i \in l} \sigma_i(t). \quad (4)$$

The macroscopic parameters are defined averaging the activities over the blocks:

$$A = \langle A_l \rangle, \quad D = \sqrt{\text{Var}(A_l)}, \quad (5)$$

where  $A$  is the usual global activity[9] and  $D$  is the block activity, respectively. One can also define the global activity of the network as  $A = (A_+ + A_-)/2$  and the block activity as  $D = (A_+ - A_-)/2$ . The network can be in the following representative phases: global activity (G, with  $A \neq 0$ ,  $D = 0$ ), block activity (B, with  $A = 0$ ,  $D \neq 0$ ) and zero activity (Z, with  $A = 0$ ,  $D = 0$ )[10]. In the next section the states of the network, in each of this phases, are presented.

### III. SIMULATION

The typical evolution of a network with  $N = 65,536$  neurons and  $K = 64$  connections (in average) for each of them, is presented in Figure 1. The probability of random connections is  $\omega = 0.127$ , and the bias for the synaptic weights is  $\bar{W} = 0.3$ . In each figure, the activities are smooth averaged over windows of  $N_w = 655$  neurons. The network starts in a noisy 2-blocks structure of activities, with initial condition  $A = 0$ ,  $D = 0.2$ .

After  $t = 12$  time steps in the network evolution, the block has been almost completed, with  $A \sim 0.0$ ,  $D \sim 0.8$ . Then a long process takes place, and at  $t = 400$  the active block is approximately filled with  $A_+ \sim 1$ , while the inactive block was destructed  $A_- \sim 0$ . In the next steps, the inactive block become attracted by the active one, and the global phase is achieved, where an active ordering is restored,  $A \sim 1$ ,  $D \sim 0$ .

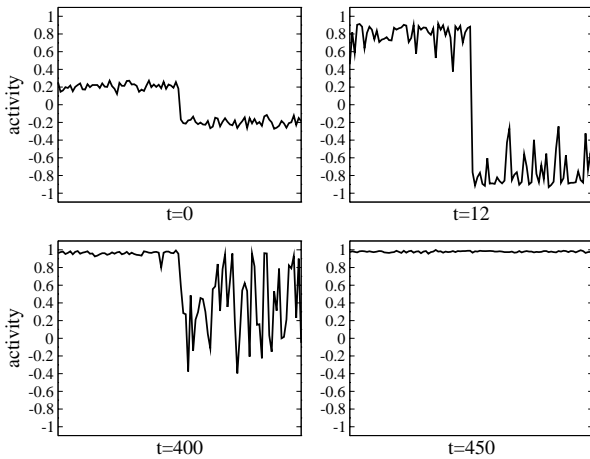


Fig. 1. Evolution of a network with  $N = 65,536$  and  $K = 64$ . The parameters are  $\omega = 0.127$  and  $\bar{W} = 0.3$ . The time steps are  $t = 0$  (top-left panel),  $t = 12$  (top-right)  $t = 400$  (bottom-left) and  $t = 450$  (bottom-right).

Both global and block activity order parameters are plotted against the time evolution in Figure 2, for the same values of the variables in the Figure 1. One sees that after an initial retrieval of the full block ordering, the network almost

suddenly (in a logarithmic time scale) crashes into a switch between B and G phases.

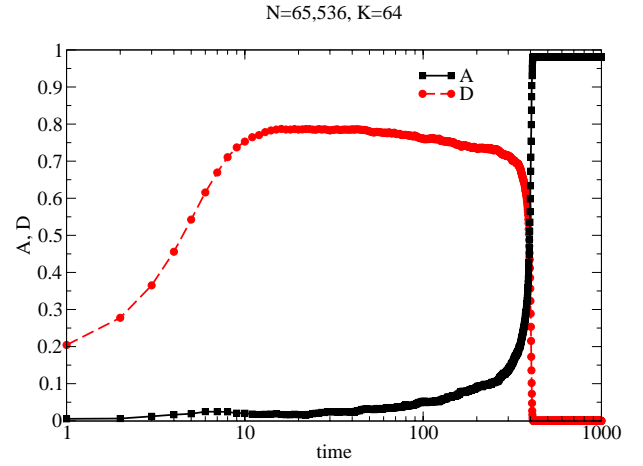


Fig. 2. Global (A) and Block (D) activity evolution in time. Stationary G phase;  $\omega = 0.127$ ,  $\bar{W} = 0.3$ . Network with  $N=65,536$  neurons and  $K=64$ .

For a somewhat smaller value of the randomness parameter, say  $\omega = 0.090$ , and keeping the other parameters the same as in Figure 2, one can observe a stable block phase. In Figure 3 the global and block activities,  $A$ ,  $D$  are plotted against the time. It is seen that up to  $t = 10,000$  time steps, the blocks doesn't change into a global ordering. The behaviour of the activity during the network evolution corresponds to the top panels in Figure 1.

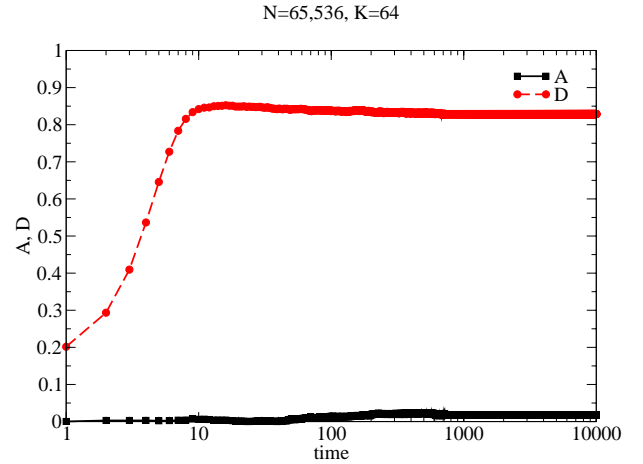


Fig. 3. Global (A) and Block (D) activity evolution in time. Stationary B phase;  $\omega = 0.090$ ,  $\bar{W} = 0.3$ . Network with  $N=65,536$  neurons and  $K=64$ .

One can describe this new phenomenon of phase transition between global and block neural activity phases, briefly, by a phase diagram. In Figure 4 it is shown in which regions of the parameters  $\omega$ ,  $\bar{W}$  there are G, B or Z stationary states. The initial condition is chosen as a noisy block phase with  $A = 0$ ,  $D = 0.2$ , and the network evolves until its attractor.

It can be concluded from this figure that block activity appears for values of  $\bar{W}$  greater than 0.16 approximately, and for increasing  $\omega$  with  $\bar{W}$ . This phase diagram has been checked against other initial conditions with  $A = 0$ ,  $D > 0$ , and it is

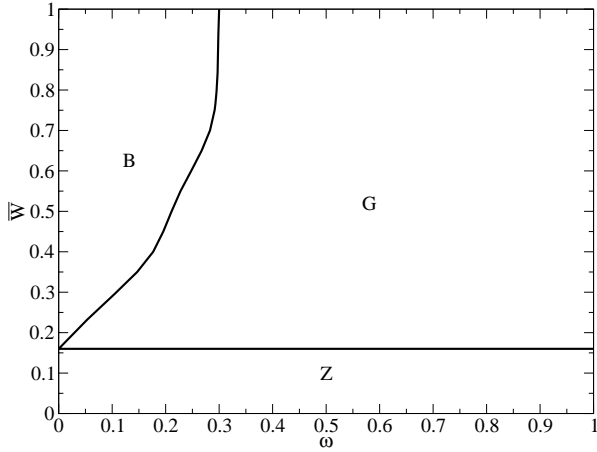


Fig. 4. Phase Diagram for block activity. B: block activity region, G: global activity region, Z: Zero region.

roughly the same. So, the B and G phases are qualitatively robust respect to all block initial conditions.

#### IV. THEORY

The simulation results presented in the previous section can be supported by a straightforward theory. The theory discussed here is based in a signal to noise ratio approximation[11]. Let the neurons be distributed within blocks  $l$ , successively with positive and negative activities,  $A_l = A_{\pm}$ . Then, following Equations (5), the block activities can be written as

$$A_l = A + y_l D, \quad (6)$$

where  $y_l \doteq \pm 1$  (according to the block) is a random variable.

The local field, Equation (1), with the Equation (3) for the synapses, can be separated in a signal and a noise terms,

$$h_i = \overline{W} K A_i + \Omega_i \quad (7)$$

where  $A_i \equiv \frac{1}{K} \sum_{j \in \{i\}} \sigma_j$ ,  $\Omega_i \equiv \sum_{j \in \{i\}} W_{ij}^T \sigma_j$  are the activity restricted to the neighbours  $\{i\}$ , and the synaptic noise, respectively.

There are local and random neighbours for each neuron, hence the signal term itself splits in localized and randomized terms, namely

$$A_i = \frac{K_L}{K} A_i^L + \frac{K_R}{K} A_i^R, \quad (8)$$

with  $A_i^{L,R} \equiv \frac{1}{K_{L,R}} \sum_{j \in L,R} \sigma_j$  where  $L$  and  $R$  are the local and random sets of neighbours, respectively, of the neuron  $\sigma_i$ .

From Equation (4), whenever the neighbours belong to a block, the localized field depends on its block activity,  $A_l$ . On the other hand, the randomized field is a sample of a global field, which does not depend on the block. Using the definition in Equation (2), one arrives to an approximation for the local field of neurons in the block  $l$ ,

$$h_l \equiv \overline{W} K [\omega A + (1 - \omega)(A + y_l D)(1 - \gamma b)] + \Omega \quad (9)$$

where the correction term  $(1 - \gamma b)$  accounts for the boundary effects between  $A_{\pm}$  blocks.

The equation for the the block-activity is then  $A_l = \langle \text{sign}(h) \rangle_{\Omega}$ , where the average in the angular brackets are over the noise  $\Omega$ . But from the Equation (6), after averaging over the  $y_l$  one gets

$$\begin{aligned} A &= \langle A_l \rangle_y = \langle \text{sign}(h) \rangle_{y,\Omega} \\ D &= \langle y A_l \rangle_y = \langle y \text{sign}(h) \rangle_{y,\Omega}, \end{aligned} \quad (10)$$

The average over  $\Omega$  stands for the noise distribution.

This noise is Gaussian distributed,  $\Omega \doteq N(0, \Delta^2)$ [2]. Its variance is given by the sum of random and local terms,  $\Delta^2 = \text{Var}(\Omega_i) = \omega \Delta_r^2 + (1 - \omega) \Delta_l^2$ . Neglecting the *feedback* terms, it is  $\Delta^2 = K$ . This approximation is valid in the limit of strongly diluted networks ( $K \ll N$ ) [12]. However, for local connections, even extreme dilution do not eliminate the feedback, and  $\Delta$  needs more precise calculations, which is outside the scope of the present work.

The continuous transition from the G to the Z phase may be analysed by taking first  $D = 0$  in the Equations (10), which gives  $A = \langle \text{sign}(\overline{W} K A + \Omega) \rangle$ , then expanding around  $A \sim 0$ . It gives the constant line:  $\overline{W} = \sqrt{\frac{\pi}{2K}}$ , which coincides with border G-Z plotted in Figure 4. The transition between B and G phases is not continuous, so no expansion is possible, but the equation:  $D = \langle \text{sign}(\overline{W} K (1 - \omega) D + \Omega) \rangle$ , is similar to the previous equation for  $A$  except that it depends on  $\omega$ . The finite solution  $D > 0$  is stable only if  $\overline{W}(1 - \omega) > \sqrt{\frac{\pi}{2K}}$ , which fits well also with the phase diagram in Figure 4.

#### V. CONCLUSION

A new type of solution, for an attractor neural network, was studied here: the block activity phase (B). When a bias in the synaptic weights is added to random weights, the network becomes ordered in a global activity phase (G), which resembles the ferromagnetic state in a spin system. This phase may coexists with a spin-glass phase, which is microscopically ordered, but without any spatial structure[8].

The B phase, however, is spatially structured: within each (mesoscopic) block, the neurons are ordered, which represents synchronization of activities in cortices of a neural system. If the connections between each block are less relevant than inside the blocks, as it is the case of small-world networks with few long-range re-wiring, the B phase is stable. If there are enough random long-range connections, the G phase attracts almost all space of configurations: even a initial condition close to a block structure leads to a final state where all neurons are ordered. One believes this B phase is robust for a whole set of topologies (for instance, power-law scaling) or neural dynamics (for instance, integrate and fire neurons), which is actually being investigated by the authors. It is also robust for learning of patterns, which was the subject of research in [13].

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