

Partitioning Ordinary Differential Equations Using Multistep Formulas

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Abstract—This paper focuses on solving Ordinary Differential Equations (ODEs) using partitioning technique known as Intervalwise Partitioning Block (IPB) method which is based on Block Backward Differentiation Formulas (BBDF) and Block Adams Formulas. Method based on BBDF requires Newton-like iteration involving the Jacobian matrix of the ODEs. These matrix operations in the iteration scheme are expensive and require a considerable amount of computational effort. Therefore, IPB technique is developed to reduce the cost of these iterations. Numerical results indicate that the IPB strategy reduce the execution time while accuracy is maintained.

Keywords—Block, ordinary differential equations, partitioning.

I. INTRODUCTION

WE shall consider the numerical solution of Ordinary Differential equations (ODEs) given by

$$y' = f(x, y) \tag{1}$$

on the interval $x \in [a, b]$, given initial values $y(a) = y_0$.

Some of the earlier works on partitioning ODEs are given in Enright and Kamel [1], Watkins and Hansonsmith [6]. Watkins and Hansonsmith developed a precise partitioning method that resembles the method proposed by Enright and Kamel but differs in that the partition of the Jacobian $\partial f/\partial y$ is partitioned into stiff and non stiff parts. Other research on partitioning are discussed by Hall and Suleiman [2], Rentrop [4], Suleiman and Baok [5] and Weiner *et al.* [7].

The partitioning technique to be developed in this paper is related to Hall *et al.* approach. Before the technique is discussed, review on block method for solving equation (1) is given.

II. OVERVIEW OF BLOCK METHOD

In two points block method, two new values, y_{n+1} and y_{n+2} are generated simultaneously at each step. Block method using Backward Differentiation Formulas (BDF) have been developed by Zarina *et al* [8] for solving the stiff parts in equation (1) and is given by

(i) constant step size

$$-\frac{1}{10}y_{n-2} + \frac{3}{5}y_{n-1} - \frac{9}{5}y_n + y_{n+1} + \frac{3}{10}y_{n+2} = \frac{6}{5}hf_{n+1}$$

$$\frac{3}{25}y_{n-2} - \frac{16}{25}y_{n-1} + \frac{36}{25}y_n - \frac{48}{25}y_{n+1} + y_{n+2} = \frac{12}{25}hf_{n+2}$$

(ii) half the step size

$$-\frac{3}{128}y_{n-2} + \frac{25}{128}y_{n-1} - \frac{225}{128}y_n + y_{n+1} + \frac{75}{128}y_{n+2} = \frac{15}{8}hf_{n+1}$$

$$\frac{2}{115}y_{n-2} - \frac{3}{23}y_{n-1} + \frac{18}{23}y_n - \frac{192}{115}y_{n+1} + y_{n+2} = \frac{12}{23}hf_{n+2}$$

(iii) increase the step size by a factor of $1.6h$

$$-\frac{208}{775}y_{n-2} + \frac{6912}{5425}y_{n-1} - \frac{13689}{6200}y_n + y_{n+1} + \frac{351}{1736}y_{n+2} = \frac{117}{124}hf_{n+1}$$

$$\frac{12544}{29875}y_{n-2} - \frac{53248}{29875}y_{n-1} + \frac{74529}{29875}y_n - \frac{2548}{1195}y_{n+1} + y_{n+2} = \frac{546}{1195}hf_{n+2}$$

The method is called Block Backward Differentiation Formulas (BBDF). BBDF method is implemented using Newton iteration which requires the solution of linear systems at each iteration with an approximate Jacobian matrix.

For solving non stiff parts in (1), the corresponding pairs of Adams type block method proposed by Majid *et. al* [3]. The method is given by

(i) constant step size

$$y_{n+1} = y_n + \frac{h}{720}(-74f_{n-2} + 11f_{n-1} + 456f_n + 346f_{n+1} - 19f_{n+2})$$

$$y_{n+2} = y_n + \frac{h}{90}(-f_{n-2} + 4f_{n-1} + 24f_n + 124f_{n+1} + 29f_{n+2})$$

(ii) half the step size

$$y_{n+1} = y_n + \frac{h}{14400}(37f_{n-2} - 335f_{n-1} + 7455f_n + 7808f_{n+1} - 565f_{n+2})$$

$$y_{n+2} = y_n + \frac{h}{90}(-f_{n-2} + 5f_{n-1} + 285f_n + 1216f_{n+1} + 295f_{n+2})$$

(iii) double the step size

$$y_{n+1} = y_n + \frac{h}{1800}(145f_{n-2} - 704f_{n-1} + 1635f_n + 755f_{n+1} - 31f_{n+2})$$

$$y_{n+2} = y_n + \frac{h}{225}(-20f_{n-2} + 64f_{n-1} + 15f_n + 320f_{n+1} + 71f_{n+2})$$

In the next section, we briefly mentioned the implementation of these methods on solving ODEs.

III. IMPLEMENTATION OF IPB

In intervalwise partitioning block technique, the whole systems of ODEs in (1) is initially treated as non stiff ODEs and solved using Adams type block method. Once there is an indication of instability due to step failure, we suspect the presence of stiffness. Once there is such an indication,

stronger tests will be imposed. The trace of the Jacobian $\partial f/\partial y$ is calculated. If the trace is negative, the whole system is changed to stiff and solved using the BBDF method. But if the trace is positive, the iteration is continued using Adams type block formula with half of the step size.

The error control algorithm is as follows:

- i. if the error control is less than tolerance limit, then the step size h is increased to gain computation speed;
- ii. in the case of step failure, the step size h is halved and the step is repeated.

IV. NUMERICAL RESULTS

In this section, some of the test problems are given to validate the efficiency of IPB technique.

Problem 1:

ODEs: $y_1' = -1002y_1 + 1000y_2^2$
 $y_2' = y_1 - y_2(1 + y_2)$
 Initial values: $y_1(0) = 1, y_2(0) = 1$
 Interval: $0 \leq x \leq 20$
 Solution: $y_1(x) = e^{-2x}, y_2(x) = e^{-x}$

Problem 2:

ODEs: $y_1' = -10y_1 + 100y_2$
 $y_2' = -100y_1 - 10y_2$
 $y_3' = -4y_3$
 $y_4' = -y_4$
 $y_5' = -0.5y_5$
 $y_6' = -0.1y_6$
 Initial values: $y_1(0) = 0.0, y_2(0) = \dots = y_6(0) = 1.0$
 Intervals: $0 \leq x \leq 20$
 $y_1(x) = e^{-10x} \sin(100x)$
 $y_2(x) = e^{-10x} \cos(100x)$
 Solution: $y_3(x) = e^{-4x}$
 $y_4(x) = e^{-x}$
 $y_5(x) = e^{-0.5x}$
 $y_6(x) = e^{-0.1x}$

The following notations are used in the tables:

- TOL : Tolerance limit
- MTD : Method employed
- TS : Total steps taken
- FA : Total number of failure steps
- IST : Total number of success steps
- MAXE : Magnitude of the maximum error of the computed solution
- NBDF : Method variable step variable order BDF
- IPB : Implementation of Intervalwise Partitioning
- TIME : The execution time in microseconds

TABLE I

NUMERICAL RESULT FOR PROBLEM 1						
TOL	MTD	FA	IST	TS	MAXE	TIME
10^{-2}	NBDF	1	22	23	1.1578e-04	0.000874
	IPB	2	24	26	1.1793e-03	0.000779
10^{-4}	NBDF	1	40	41	8.9855e-06	0.001068
	IPB	2	43	45	2.8144e-05	0.000956
10^{-6}	NBDF	6	85	91	6.8684e-08	0.001566
	IPB	3	97	100	5.4695e-06	0.001461

TABLE II

NUMERICAL RESULT FOR PROBLEM 2						
TOL	MTD	FA	IST	TS	MAXE	TIME
10^{-2}	NBDF	1	99	100	1.1066e-02	0.008298
	IPB	1	102	103	3.1328e-02	0.006591
10^{-4}	NBDF	1	262	263	1.0812e-04	0.014621
	IPB	3	258	261	4.2225e-04	0.013554
10^{-6}	NBDF	4	656	660	1.0171e-06	0.030956
	IPB	6	654	660	1.0177e-06	0.029394

V. CONCLUSION

The results shows that despite the larger number of steps using IPB, the integration at all tolerance is much cheaper in total execution time than the integration without partitioning. In conclusion, we have demonstrated that it is favourable to partition ODEs rather than treat the systems as a stiff system to all equations.

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REFERENCES

- [1] Enright, W.H. and Kamel, M.S., *Automatic Partitioning of Stiff Systems and Exploiting the Resulting Structure*, ACM Transactions on Mathematical Software, 1979, Vol 5, No 4, 374-385.
- [2] Hall, G. and Suleiman, M.B., *A single code for the solution of stiff and nonstiff ODEs*, SIAM J. Stat. Computing, 1985, 6(3):684-697.
- [3] Majid, Z., *Parallel Block Methods For Solving Ordinary Differential Equations*, PhD Thesis, Universiti Putra Malaysia
- [4] Rentrop, P., *Partitioned Runge-Kutta Method with stiffness detection and stepsize control*, Numer. Math, 1985, 47: 545-564.
- [5] Suleiman, M.B. and Baok, S., *Using nonconvergence of iteration to partition ODEs*, Applied Math and Computation, 1992, 49: 111-139.
- [6] Watkins, D.S. and Hansonsmith, R.W., *The Numerical Solution of Separably Stiff Systems by Precise Partitioning*, ACM Transactions on Mathematical Software, 198, Vol 9, No 3, 293-301.
- [7] Weiner, R., Arnold, M., Rentrop, P. and Strehmel, K., *Partitioning strategies in Runge-Kutta type methods*, IMA Journal of Numerical Analysis, 13: 303-319.
- [8] Zarina Bibi, I., Khairil Iskandar, O., Suleiman, M., 2007. *Variable Stepsize Block Backward Differentiation Formula For Solving Stiff ODEs*, Proceedings of World Congress on Engineering 2007, London, U.K. Vol II: pg 785-789. ISBN: 978-988-98671-2-6