# The Root System: A Dynamic Coordinate System

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***Abstract*****—**The imaginary Argand plane and the Cartesian coordinate system are two different systems which can not be linked into a common system without modifications. This means that complex and imaginary numbers can not be depicted in the ordinary Cartesian coordinate system but only in a separate plane, the Argand plane. This paper is describing a proposal for a combined system, which is derived from a modified Cartesian coordinate system, combined with a local real and imaginary plane on the axes. It is an orthogonal system in the Euclidian space.

The Cartesian system is changed to root level in the way that it gets a real and an imaginary side. In principle by taking the square root out of the ordinary coordinate axis of the Cartesian coordinate system. This means that the positive axis will be the real side and the negative axis the imaginary side in the proposed system. In this paper for brevity the system will be called the Root System.

A local coordinate plane with real coordinates on the imaginary axes and imaginary coordinates on the real axes will be mounted at right angle to the axes of the Root system. They are bounded by a circle/ellipse with the radius equal to its location along the coordinate axis from origo. The proposed system is forming a conical system of circular/elliptical cones with a central coordinate axis which is at the root level of the Cartesian system.

The system has six different coordinate axes in three dimensions a real and an imaginary for each axis in the Cartesian system. The end of the cones at defined coordinate will be called the windows. The system is dynamic which means that it is not static but is affected by factors outside of the system. The axes in the windows of the cones are graded in radians, which mean that the coordinates of the windows can be written with trigonometric or hyperbolical functions.

In physics is the Root system a local system attached to the object and function as a communicator to the surrounding space. The system adjusts it self depending on the surrounding conditions. For example an object travelling with a speed near to the speed of light the space for the object will change by contraction of the corresponding axis.

In this paper, is demonstrated how to solve and depict the geometric solution for linear, square, cubic and quartic equations with the proposed system. In addition, it is also showed how the system can be used to display the solution of the Lorentz contraction, and other physical cases.[4], [5], [6], [7]

***Keywords*****—**Dynamic coordinate system, conical system, coordinate system, Lorentz contraction, root system, Hilbert's space, Dirac's equation, MWI

## I. Introduction

IN the textbook Complex Analysis with *Mathematica* [8] is an example of a complex equation with complex coefficients, which can not be depicted in the Cartesian coordinate system. This equation has been a guiding star to find a graphical and geometrical system to depict the solution of such an equation in the number space.

In the book Riemann's Zeta Function [1] the author suggest that one should read the classics and beware of secondary sources. These suggestions are taken ad notam. Two classics have been important for this work, René Descartes [2] for the study of the quadratic and Omar Khayyam [3] for the cubic equations. It has been interesting and fascinating to redo these mathematicians work with *Mathematica* nearly 1000 years after Omar Khayyam's death.

In the proposed system, known as the Root system in this document, René Descartes method of solving quadratic equations geometrical have been used as a base for the proof of quadratic equations in the Root system. Omar Khayyam's work has been important for the cubic equation. With the help of their works a method has been developed to depict the geometric and graphical solution for real, imaginary and complex roots of linear, quadratic, cubic and quartic equations. The hyperbolic and trigonometric functions are an important part of this system as the axes for the windows are graded in radians.

The goal for this paper is to create a system that combines the imaginary plane and the Cartesian coordinate system to a combined system.

## II. Coordinate System

### A. The Cartesian system

The Cartesian system needs no further explanation Fig. (2.1)

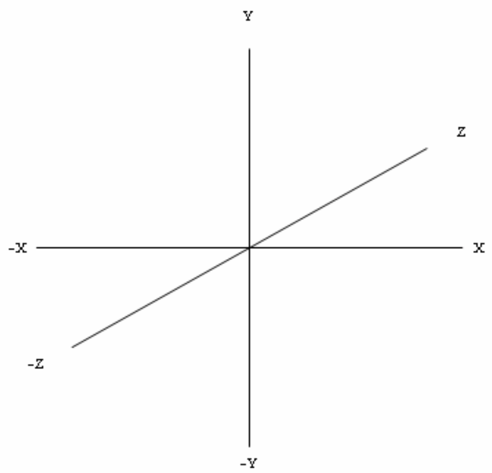



Fig. 2.1 the Cartesian system

*B. The Argand Plane*

The real and imaginary parts of complex numbers, can be regarded as coordinates in a two dimensional plane, the Argand plane commonly known as Argand plane Fig (2.2).

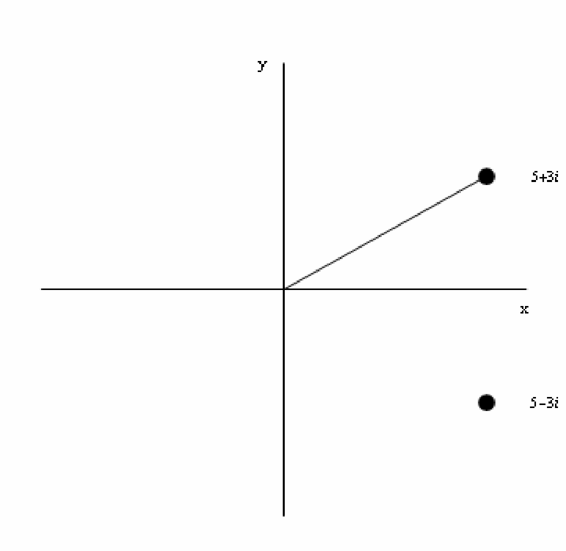



Fig. 2.2 the Argand plane

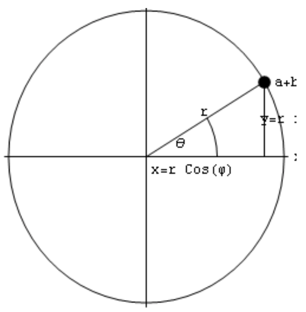
The complex number *x+iy* are regarded as the point with coordinates (*x, y*) and all complex number C are indentified with the two –dimensional plane $R^2$. The real part is the x-coordinate and the imaginary part the y-coordinate.

*C. The polar complex plane*

A complex plane in its Cartesian form, Fig. (2.3)

$z = x + iy$

Instead of using the Cartesian system, it is possible to use polar coordinates (r, ϕ) , related to Cartesian coordinates as




$$x = Cos\phi$$
$$y = iSin\phi$$
$$z = r(Cos\phi + iSin\phi)$$

Fig. 2.3 The complex plane in polar form

## III. THE ROOT SYSTEM

The real Cartesian coordinate system, and the complex Argand plane, can not be combined and therefore the Cartesian system must be changed in such a way, that it gets an imaginary and a real side. One way to achieve this is to take the square root out of the coordinate axes of the system. The positive axes will represent the real numbers and the negative axes the imaginary numbers. The axes are graded in the same way as for the ordinary Cartesian system but it is only representing real or imaginary numbers. The revised Cartesian system is depicted in Fig. (3.1).

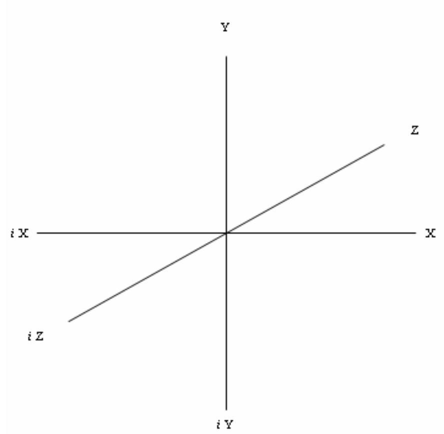



Fig. 3.1 the Root system

To combine the Root system and a revised complex plane, the complex planes have to be mounted orthogonal to the axis of the revised Cartesian coordinate system, the Root system. The combined system has the radius of the polar plane equal to the value of the coordinate of the Root system's coordinate axis. The coordinate axes of the Root system will be surrounded by cones around the axes of the Root system.

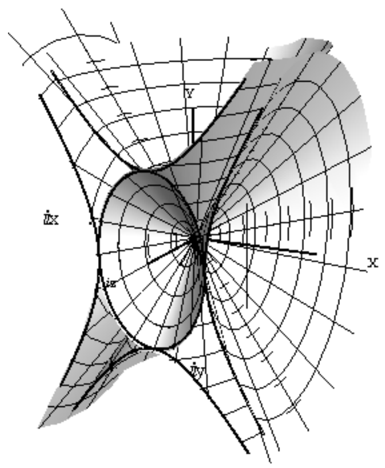



Fig. 3.2 A 3-dimensional view of the Root system

If we make a cut through an axis Fig (3.2) we will get a circular/elliptic cut through the axis and hyperbolas from the cut through the adjacent axes. This cut is called the window of the axis. We get two types depending on if the axis is real or imaginary. For the real axis the surrounding cone will be real and for the window the circle/ellipse, radius and angle will even be real but the coordinates (3.2) inside the window will be imaginary Fig.(3.3)

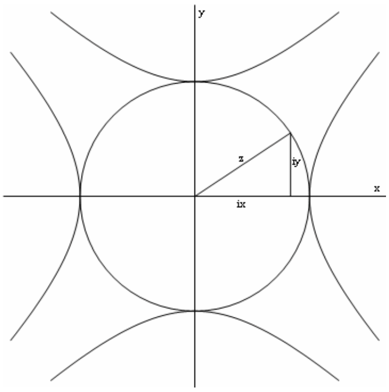



Fig. 3.3 the window of the real axis

For the imaginary axis the cone will even be imaginary and so even the circle/ellipse, radius and angle but the coordinates (3.1) will be real Fig. (3.4)

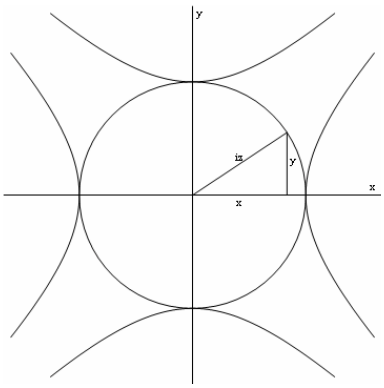



Fig. 3.4 the window of the imaginary axis

For the windows of the imaginary z-axis the following linkage is valid Fig (3.3)

$$\begin{cases} x^2+y^2+(iz)^2=0 \\ y^2+z^2+(ix)^2=0 \\ z^2+x^2+(iy)^2=0 \end{cases} \Rightarrow \begin{cases} x^2+y^2=z^2 \\ y^2+z^2=x^2 \\ z^2+x^2=y^2 \end{cases} \tag{3.1}$$

And for the real z-axis Fig (3.4):

$$\begin{cases} (ix)^2+(iy)^2+z^2=0 \\ (iy)^2+(iz)^2+x^2=0 \\ (iz)^2+(ix)^2+y^2=0 \end{cases} \Rightarrow \begin{cases} x^2+y^2=z^2 \\ y^2+z^2=x^2 \\ z^2+x^2=y^2 \end{cases} \tag{3.2}$$

There is a dualism in the equations (3.1), (3.2) as it is not possible to differentiate if it is real or imaginary coordinates as both gives the same result. If we add together the three equations we get the following linkage:

$$x^2+y^2+z^2=0 \tag{3.3}$$

Equation (3.3) is the equation for the Root system. To be able to use this system for algebraic equations it must be possible to influence every coordinate with a coefficient which can be a real, an imaginary or a complex number. The Root system will be influenced by these coefficients, which mean that the system is not static, but dynamic and is changing depending on the kind, and the value of the coefficients. Equation (3.3) will change to:

$$a^2y^2+b^2x^2+c^2z^2=0 \tag{3.4}$$

The equations for the windows of the imaginary z-axis are then:

$$\begin{cases} b^2x^2+a^2y^2+c^2(iz)^2=0 \\ a^2y^2+c^2z^2+b^2(ix)^2=0 \\ c^2z^2+b^2x^2+a^2(iy)^2=0 \end{cases} \Rightarrow \begin{cases} b^2x^2+a^2y^2=c^2z^2 \\ a^2y^2+c^2z^2=b^2x^2 \\ c^2z^2+b^2x^2=a^2y^2 \end{cases} \tag{3.5}$$

If the coordinates are complex the following connection must be fulfilled for all the axes.

$$\begin{aligned} &\mathrm{Re}(a^2y^2)+\mathrm{Re}(b^2x^2)+\mathrm{Re}(c^2z^2)=0 \\ &\mathrm{Im}(a^2y^2)+\mathrm{Im}(b^2x^2)+\mathrm{Im}(c^2z^2)=0 \end{aligned} \tag{3.6}$$

This means that the absolute value of the coordinates of the axes must be equal to zero

$$|a^2y^2|+|b^2x^2|+|c^2z^2|=0 \tag{3.7}$$

And the equations for the real window will change in the same manner.

In this paper the z-axis will be used for the known value, and the x-axis as variable, and the y-axis as the depending variable. Equation (3.4) will be rewritten to (3.8)

$$y^2+k^2x^2+z^2=0 \tag{3.8}$$

This relationship determines that for the Root system that the sum of an equation is always three-dimensional and the sum is always zero (3.6) - (3.8). For polar plane on the imaginary z-axis, Fig (3.3), z is imaginary and radius $|z|$ and the local coordinates are real. The same is the case for the real z-axis Fig (3.4). The radius of the polar plane is real and equal to |z| and the local coordinates imaginary. This means that on the imaginary part of the global system, there is an imaginary cone around the coordinate axis and around the real coordinate axis there is a real cone. The other axes are built up in the same way. The coefficient *k* and the known value *z* can take any possible value.

From Fig (3.5) the following equation is derived:

$$\begin{aligned} &\frac{z-y}{kx}=\frac{kx}{z+y}\Rightarrow k^2x^2=z^2-y^2 \\ &y^2+k^2x^2-z^2=0 \end{aligned} \tag{3.9}$$

If *z* is imaginary the equation change to:

$$\begin{aligned} &\frac{iz-y}{kx}=\frac{kx}{iz+y}\Rightarrow k^2x^2=-z^2-y^2 \\ &y^2+k^2x^2+z^2=0 \end{aligned} \tag{3.10}$$

The above expression is identical with (3.8) for the imaginary z-axis, and Fig 3.5 is in agreement with the windows of the axes.

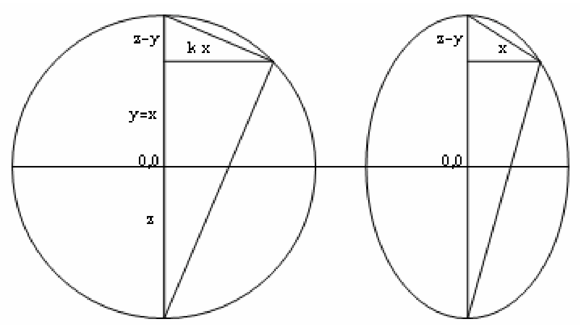



Fig. 3.5 the base for the Root system for linear equation with real coefficient *k*

If we change the coefficient *k* to an imaginary value as in Fig (3.6) which is in agreement with (3.8, 3.9) with an imaginary value of *k*

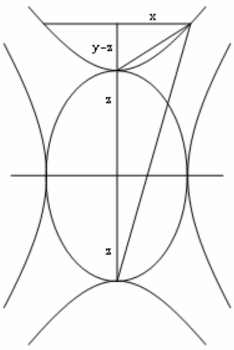



Fig. 3.6 the base for the Root system for quadratic equation with imaginary coefficient *k*

In line with the derivations of the Root system, Fig (3.5-3.6) the obtained solution for real or imaginary values of the equations are either on the ellipse or on the hyperbola. For complex solutions the absolute value lies on the function line between the hyperbola and the ellipse. Imaginary values appear on the hyperbola for the imaginary z-axis and on the ellipse for real values (3.1). For the real z-axis, it is the opposite, imaginary values on the ellipse and real on the hyperbola (3.2).

## IV. Hyperbolic Functions

Hyperbolic functions are important for the Root System as hyperbolic curves are involved in the solving of equations. A summary of functions regarding the hyperbolic radians is therefore adequate.

The geometrical definition of the radian appears from Fig 4.1. Let O be the origo in the figure, A the vertex and P any point on the branch B'AB of the rectangular hyperbola $x^2$-$y^2$=$a^2$ with the coordinates *(x,y)*. Set *OM=x, MP=y, OA=a.* The shaded area in Fig.4.1 is given by:

$$OPAP' = a^2 \log_e \frac{x+y}{a}$$

If the angle POP', in hyperbolic radian is denoted by *u*

Hyperbolic radians

$$u = \frac{areaOPAP'}{a^2}$$

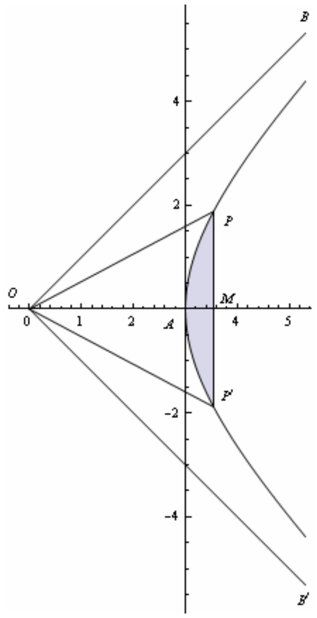



Fig. 4.1 hyperbolic radians

$$Sinh(u) = \frac{y}{a}$$

$$Cosh(u) = \frac{x}{a}$$

Exponential - definition

$$Sinh(u) = \frac{e^u - e^{-u}}{2}$$

$$Cosh(u) = \frac{e^u + e^{-u}}{2}$$

$$Tanh(u) = \frac{e^u - e^{-u}}{e^u + e^{-u}}$$

### A. Hyperbolic Functions of Complex Argument

The exponential- function can even be determined for complex arguments and so even the hyperbolic functions. The relations to the ordinary trigonometric functions are given by Euler's formula for complex numbers.

$$e^{ix} = \cos x + i \sin u$$

$$Sinh(iu) = i \sin u$$

$$Cosh(iu) = \cos u$$

$$Tanh(u) = i \tan u$$

### B. Inverse hyperbolic functions

The inverse hyperbolic functions are the area hyperbolic functions. The name leads to the fact that they calculate the area of a sector of the rectangular hyperbola $x^2$-$y^2$=$a^2$.

In the same way as inverse trigonometric functions calculate the arc length of a sector of the unit circle.

$$ArSinh(x) = \log_e (x + \sqrt{x^2 + 1}$$

$$ArCosh(x) = \log_e (x + \sqrt{x^2 - 1}$$

$$ArTanh(x) = \log_e \frac{1+x}{1-x}$$

## V. Linear Equations

### A. Linear equations

To be able to use (3.8) for solving linear equations the equation has to be expressed in another way. The common way to set up a linear equation is to express the equation as (5.1a)

$$y - a + b(x - c) + d = 0 \qquad (5.1a)$$

By transforming it to:

$$y + bx + (d - a - bc) = 0 \qquad (5.1b)$$

And substitute the known values to $z$ equal to $\sqrt{d-a+bc}$ and $k=\sqrt{b}$ the expression will be three dimensional and can be expressed as:

$$y^2 + k^2x^2 + z^2 = 0 \tag{5.2}$$

This is in agreement with (3.8) if z is real. If $z$ is imaginary the geometrical expression will be on the window of the imaginary $z$ axis and if it is real on the window of the real $z$ axis. The expression for the function of the Root system will always originate from origo and for a linear equation will be expressed as:

$$y = \alpha x \tag{5.3}$$

The result from these equations are on the root level and the values for the coordinates has to be squared to get the result as in the normal Cartesian system (3.3)

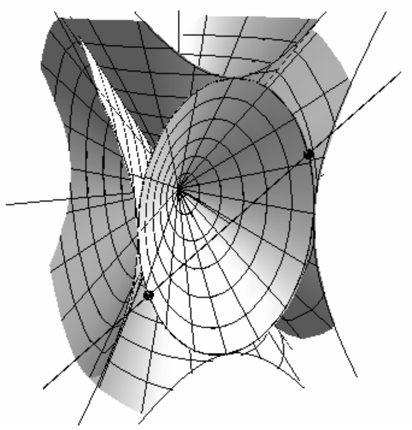
*B. Example of geometrical solutions for linear equations*

Geometrical plots for each of the z-axes will be shown in the following examples. The first five will show equations with imaginary and real $z=3$ and the function $y=0.75x$ and the last examples with a complex coefficient. Lastly examples with complex z-value and complex coefficient are solved and plotted

*1) Imaginary Window*

In this part the results from different coefficients of the equation will be depicted. The arithmetic and trigonometric calculation are shown. If $k>1$ and real the roots of the equation (5.1) will be on the ellipse Fig (5.1) with real values and if $k$ is imaginary on the hyperbola of the x-axis Fig (5.2). With imaginary values, if $k<1$ and imaginary, the roots will fall on the ellipse Fig 5.3 $k$ with real values on the y-axis with real roots Fig.5.4. If the axes are graded in radians then $k$ is equal to a $\cot(\phi)$ .

*Example 5A*



$z = 3i,\ k = 1.5$

*Aritmetric*

$\alpha = 0.75$

$(\alpha x)^2 + 1.5^2x^2 - 3^2 = 0$

$x = \pm 1.78885, y = \pm 1.34164$

*Trigonometric* :

$r = |z| = 3$

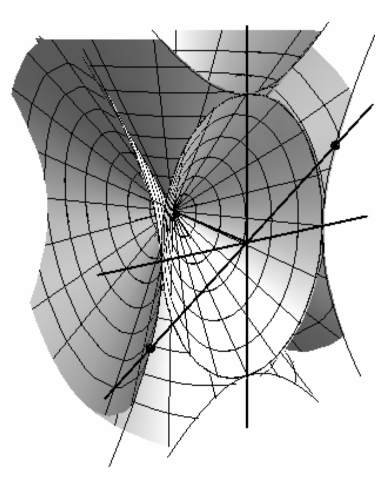
$\phi = ArcTan\dfrac{\alpha}{k}$

$x = rSin(\phi), x = \pm 1.78885$

$y = \dfrac{r}{k}Cos(\phi), y = \pm 1.34164$

Fig. 5.1 Example 5A, imaginary window, *z=3i, k=1.5*

*Example 5B*



$z = 3i,\ k = 1.5i$

*Aritmetric*

$\alpha = 0.75$

$(\alpha x)^2 - 1.5^2x^2 - 3^2 = 0$

$x = \pm 2.3094i, y = \pm 1.73205i$

*Trigonometric* :

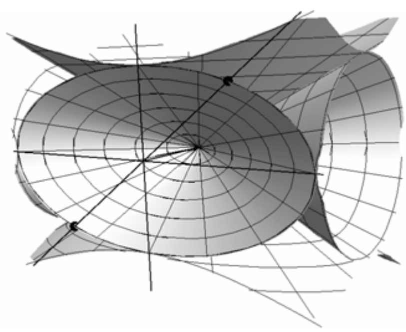
$\phi = ArcTan\dfrac{\alpha}{k}$

$x = zSin(\phi), x = \pm 2.3094i$

$y = \dfrac{z}{k}Cos(\phi), y = 1.73205i$

Fig. 5.2 Example 5B, imaginary window, *z=3i, k=1.5i*

*Example 5C*



$z = 3i,\ k = 0.5$

*Aritmetric*

$\alpha = 0.75$

$(\alpha x)^2 + 0.5^2x^2 - 3^2 = 0$

$x = \pm 3.3282, y = \pm 2.49615$

*Trigonometric* :

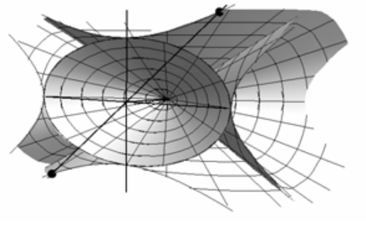
$\phi = ArcTan\dfrac{\alpha}{k}$

$x = zSin(\phi), x = \pm 3.3282$

$y = \dfrac{z}{k}Cos(\phi), y = 2.49615$

Fig. 5.3 Example 5C, Imaginary window *z=3i, k=0.5*

*Example 5D*



$z = 3i,\ k = 0.5i$

*Aritmetric*

$\alpha = 0.75$

$(\alpha x)^2 - 0.5^2x^2 - 3^2 = 0$

$x = \pm 5.36656, y = \pm 4.02492$

*Trigonometric* :

$\phi = ArcTan\dfrac{\alpha}{k}$

$x = zSin(\phi), x = \pm 5.36656$

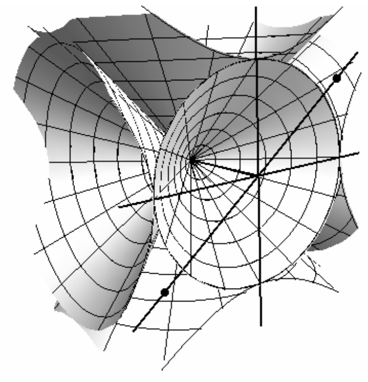
$y = \dfrac{z}{k}Cos(\phi), y = \pm 4.02492$

Fig. 5.4 Example 5D, Imaginary window, *z=3i, k=0.5i*

*Complex coefficients*

If the coefficient is complex the absolute value of the roots will neither be on the ellipse nor on the hyperbola but on the function line between. The position of the root depends on the proportion between the real and imaginary part of the complex number. If the real part is greater the root will be closer to the ellipse and if the imaginary part is greater closer to the hyperbola Fig (5.5). The geometrical picture of the complex root has to be shown in a separate graph with the squared result in accordance with (3.1), (3.6), the windows of the three axes are put together in Fig (5.6). The sum of the imaginary and real part has to be zero.

*Example 5E*



$z = 3i,\ k = 0.6 + 0.7i$

*Aritmetric*

$\alpha = 0.75$

$(\alpha x)^2 + (0.6+0.7i)^2 x^2 - 3^2 = 0$

$x = \pm 2.63499 \mp 1.60705i$

$y = \pm 1.97624 \mp 1.20528i$

*Trigonometric*:

$\phi = ArcTan\frac{\alpha}{k}$

$x = zSin(\phi),\ x = \pm 2.63499 \mp 1.60705i$

$y = \frac{z}{k}Cos(\phi),\ y = \pm 1.97424 \mp 1.20528i$

Fig. 5.5 Example 5E, Imaginary window *z=3i, k=0.6+0.7i*

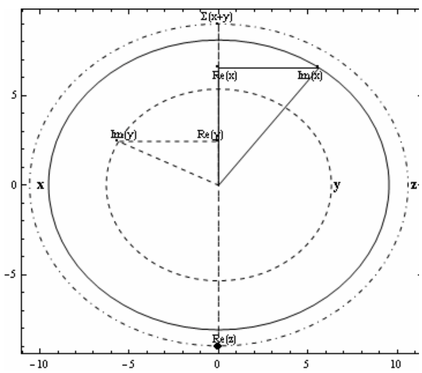
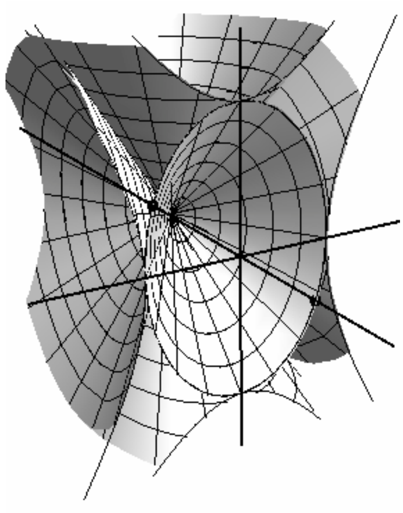



Fig. 5.6 Expanded roots example 5E

*Real Window*

In the same way as for the imaginary axis the same equations are solved but with a real value for *z*. If *k* in (5.2) is real the roots of the equation will be on the ellipse Fig (5.7) and if imaginary on the hyperbola Fig (5.8) in contrary to the imaginary window. As for the imaginary window the results from six different coefficients of the equation will be depicted. If *k>1* the roots will fall on the hyperbola of the x-axis Fig (5.8) and if it is imaginary with real values on the ellipse of the z-axis Fig.(5.7). If *k<1* and imaginary the roots will fall on the hyperbola of the y-axis else on the ellipse of the z-axis. The values of the roots are imaginary on both the ellipse and hyperbola.

*Example 5F*



$z = 3,\ k = 1.5$

*Aritmetric*

$\alpha = 0.75x$

$(\alpha x)^2 + 1.5^2 x^2 + 3^2 = 0$

$x = \pm 1.78885i, y = \pm 1.34164i$

*Trigonometric*:

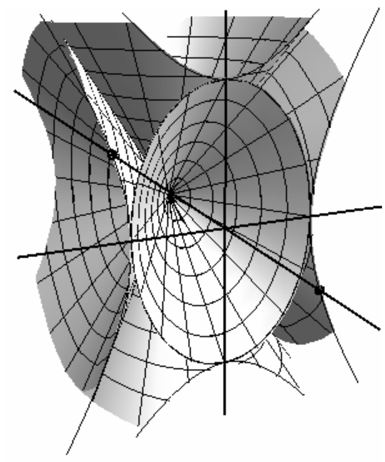
$\phi = ArcTan\frac{\alpha}{k}$

$x = zSin(\phi),\ x = \pm 1.78885i$

$y = \frac{z}{k}Cos(\phi),\ y = \pm 1.34164i$

Fig. 5.7 Example 5F, Real window *z=3, k=1.5*

*Example 5G*



$z = 3,\ k = 1.5i$

*Aritmetric*

$\alpha = 0.75x$

$(\alpha x)^2 - 1.5^2 x^2 + 3^2 = 0$

$x = \pm 2.3094, y = \pm 1.73205$

*Trigonometric*:

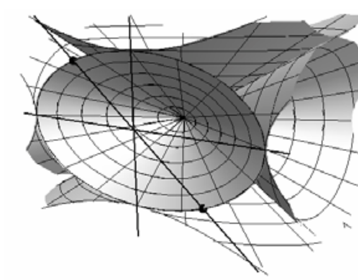
$\phi = ArcTan\frac{\alpha}{k}$

$x = zSin(\phi),\ x = \pm 2.3094$

$y = \frac{z}{k}Cos(\phi),\ y = \pm 1.73205$

Fig. 5.8 Example 5G, Real window, *z=3, k=1.5i*

*Example 5H*



$z = 3,\ k = 0.5$

*Aritmetric*

$\alpha = 0.75$

$(\alpha x)^2 + 0.5^2 x^2 + 3^2 = 0$

$x = \pm 3.3282i, y = \pm 2.49615i$

*Trigonometric*:

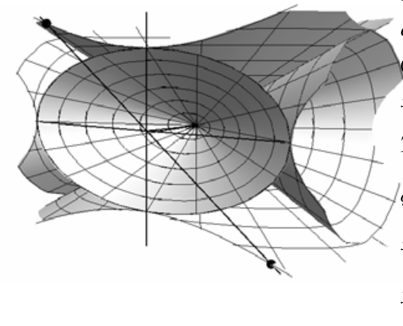
$\phi = ArcTan\frac{\alpha}{k}$

$x = zSin(\phi),\ x = \pm 3.3282i$

$y = \frac{z}{k}Cos(\phi),\ y = \pm 2.49615i$

Fig. 5.9 Example 5A, Real window *z=3, k=0.5*

*Example 5I*



$z = 3,\ k = 0.5i$

*Aritmetric*

$\alpha = 0.75$

$(\alpha x)^2 - 0.5^2 x^2 + 3^2 = 0$

$x = \pm 5.36656i, y = \pm 4.02492i$

*Trigonometric*:

$\phi = ArcTan\dfrac{\alpha}{k}$

$x = zSin(\phi), x = \pm 5.36656i$

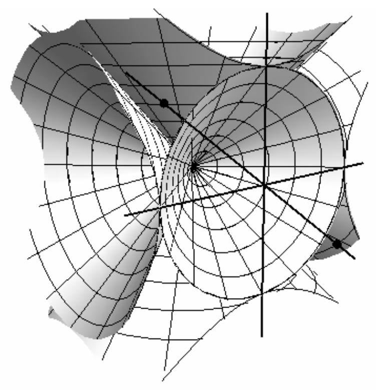
$y = \dfrac{z}{k}Cos(\phi), y = \pm 4.02492i$

Fig. 5.10, Example 5I, Real window *z=3, k=0.5i*

*2) Complex Coefficient*

In the same way as for the coefficient for the imaginary window if it is complex the absolute value of the roots will neither be on the ellipse nor on the hyperbola but on the function line between. But opposite to the imaginary window if the real part is greater the root will be closer to the hyperbola and if the imaginary part is greater closer to the ellipse Fig (5.11). The geometrical picture of the complex roots are shown in a separate graph with the squared result in accordance with (3.1), (3.6), Fig (5.12).The sum of the imaginary and real part will balance the z value.

*Example 5J*



$z = 3,\ k = 0.6 + 0.7i$

*Aritmetric*

$\alpha = 0.75$

$(\alpha x)^2 + k^2 x^2 + 3^2 = 0$

$x = \pm 1.60705 + 2.63499i$

$y = \pm 1.20528 + 197624i$

*Trigonometric*:

$\phi = ArcTan\dfrac{\alpha}{k}$

$x = zSin(\phi), x = \pm 1.60705 \pm 2.63499i$

$y = \dfrac{z}{k}Cos(\phi), y = \pm 1.20528 + 1.97624i$

Fig. 5.11 Example 5J, Real window *z=3, k=0.6+0.7i*

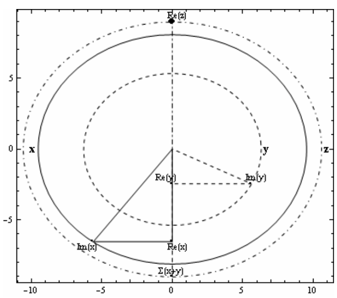
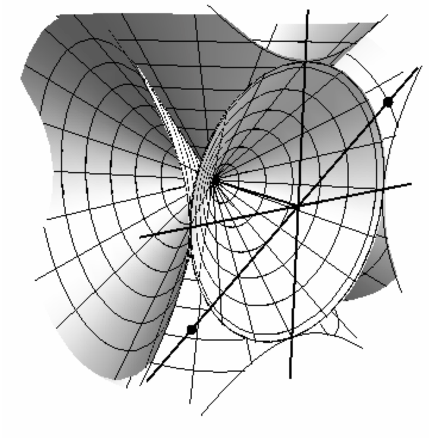



Fig. 5.12 Expanded complex roots Example 5J

*3) Complex coefficients and complex z*

In the same way as for complex coefficients an equation with both complex coefficient and complex value of *z* will be plotted in the same manner Fig (5.13), Fig (5.14)

*Example 5K*



$z = 2 + 2.5i$

$k = 0.7 + 0.8i$

*Aritmetric*

$\alpha = 0.75$

$(\alpha x)^2 + k^2 x^2 + z^2 = 0$

$x = \pm 0.829838 \mp 2.81055i$

$y = \pm 0.622379 + 2.10792i$

*Trigonometric*:

$\phi = ArcTan\dfrac{\alpha}{k}$

$x = zSin(\phi),$

$x = \pm 0.829838 \mp 2.81055i$

$y = \dfrac{z}{k}Cos(\phi),$

$y = \pm 0.622379 \mp 2.10792i$

Fig. 5.13, Example 5K, Complex window, *z=2+2.5i, k=0.7+0.8i*

The result in Fig. 5.13 is the absolute value for the axes (3.7). To get the actual values for the variables the results has to be squared (3.8). The result is showed in Fig. 5.14

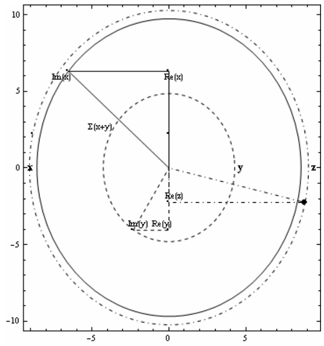



Fig. 5.14 Expanded complex roots, Example 5K

## VI. Quadratic Equations

René Descartes [2] was a master in solving equations geometrical with a ruler and a compass. He solved a quadratic equation (6.1) by lines and a circle on a plane surface.

$$z^2 = az + b^2 \qquad (6.1)$$

In this section René Descartes method will be used as a base to develop a geometrical solution for solving quadratic equations with the Root system.

### *A. René Descartes*

First René Descartes proofs with his own words:

If we have an equation, we construct a right triangle NLM, Fig. (6.1), with one side LM, equal to b, the square root of the known quantity $b^2$ and the other side LN equal to ½ a, that is, to half the other know quantity which was multiplied by z which I supposed to be the unknown line. The prolonging MN, the hypotenuse of this triangle, to O so that NO is equal to NL, the whole line OM is the required line z. This is expressed in the following way:

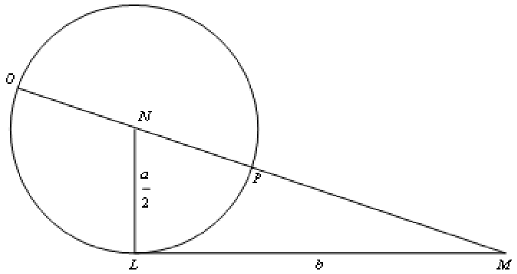



Fig. 6.1 René Descartes geometrical solution positive roots

$$z = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2} \tag{6.2}$$

But if we have $y^2 = ay + b^2$, where $y$ is the quantity whose value is desired, we construct the same right triangle NLM, and the hypotenuse MN lay off NP, and the reminder PM is $y$, the desired root.

Those I have

$$y = -\frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2} \tag{6.3}$$

In the same way, if I had $x^4 = -ax^2 + b^2$, PM would be $x^2$

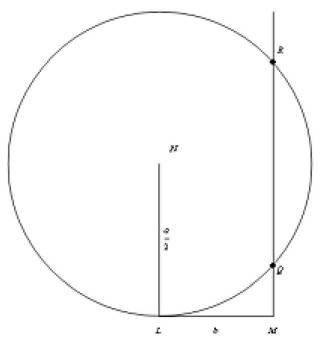
And I should have

$$x = \sqrt{-\frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2}} \tag{6.4}$$

And so on for other cases.

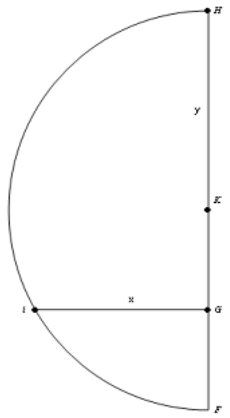
Finally, if we have $z^2 = az - b^2$, we make NL, Fig. (6.2) equal to ½a and LM equal to b as before; then, instead of joining the points M and N, I draw MQR parallel to LN, and with N as a center describe a circle through L cutting MQR in the point Q and R; then z, the line sought, is either MQ or MR, for in this case it can be expressed in two ways, namely:

And if the circle described about N and passing through L neither cuts nor touches the line MQR, the equation has no root, so that we may say that the construction of the problem is impossible.



$$z = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 - b^2}$$
$$z = \frac{1}{2}a - \sqrt{\frac{1}{4}a^2 - b^2} \tag{6.5}$$

Fig. 6.2 René Descartes geometrical solution for $z^2 = az - b^2$



If the square root of GH is desired, Fig (6.3), I add, along the same straight line, FG equal to unity; then, bisecting FH at K, I describe the circle FIH about K as a center, and draw from G a perpendicular and extend it to I, and GI is the required root.

Fig. 6.3 René Descartes geometrical solution for to get the square roots of a number

*B. The Root system*

Quadratic equations can be derived in the same way as for a linear equation out of a right angle circumscribed by a circle but with $y = x^2$, Fig.(6.4), and Fig.(6.5) for imaginary values

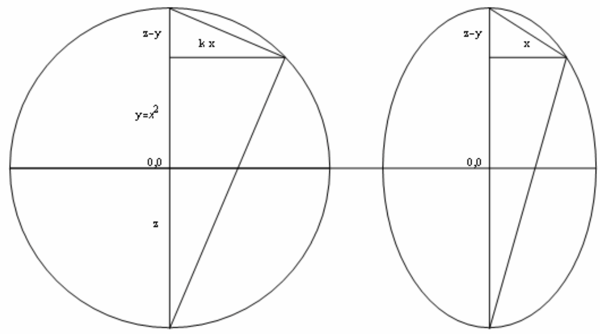



Fig. 6.4 the base for the Root system for quadratic equation with real coefficient *k*

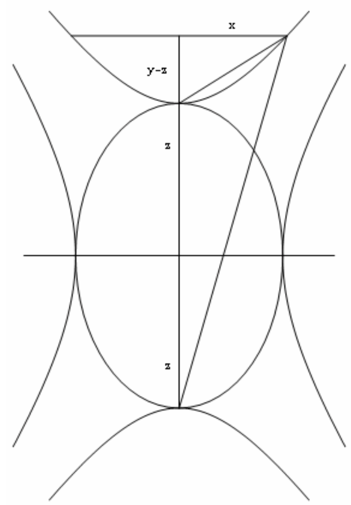



Fig. 6.5 the base for the Root system for quadratic equation with imaginary coefficient *k*

Fig 6.4 gives:

$$\frac{z-x^2}{kx}=\frac{kx}{z+x^2}$$

$$(z-x^2)(z+x^2)=k^2x^2 \tag{6.6}$$

for real z:

$$x^4+k^2x^2-z^2=0$$

for imaginary z:

$$x^4+k^2x^2+z^2=0$$

There are five kinds of solutions to quadratic equations:
1. Real coefficient k and real *z*
2. Real coefficient k and imaginary *z*
3. Imaginary coefficient k and real *z*
4. Imaginary coefficient k and imaginary *z*
5  Complex coefficient and complex *z*.

*1) Real coefficient and Imaginary z=ic*

$$y=-\frac{1}{2}k^2+\sqrt{\frac{1}{4}k^4+c^2}$$
$$y=-\frac{1}{2}k^2-\sqrt{\frac{1}{4}k^4+c^2}$$
$$x=\pm\sqrt{-\frac{1}{2}k^2+\sqrt{\frac{1}{4}k^4+c^2}} \tag{6.7}$$
$$x=\pm\sqrt{-\frac{1}{2}k^2-\sqrt{\frac{1}{4}k^4+c^2}}$$

*2) Real coefficient and real z=c*

$$y=-\frac{1}{2}k^2+\sqrt{\frac{1}{4}k^4-c^2}$$
$$y=-\frac{1}{2}k^2-\sqrt{\frac{1}{4}k^4-c^2}$$
$$x=\pm\sqrt{-\frac{1}{2}k^2+\sqrt{\frac{1}{4}k^4-c^2}} \tag{6.8}$$
$$x=\pm\sqrt{-\frac{1}{2}k^2-\sqrt{\frac{1}{4}k^4-c^2}}$$

*3) Imaginary coefficient and imaginary z=ic*

$$y=\frac{1}{2}k^2+\sqrt{\frac{1}{4}k^4+c^2}$$
$$y=\frac{1}{2}k^2-\sqrt{\frac{1}{4}k^4+c^2}$$
$$x=\pm\sqrt{\frac{1}{2}k^2+\sqrt{\frac{1}{4}k^4+c^2}} \tag{6.9}$$
$$x=\pm\sqrt{\frac{1}{2}k^2-\sqrt{\frac{1}{4}k^4+c^2}}$$

*4) Imaginary coefficient and real z=c*

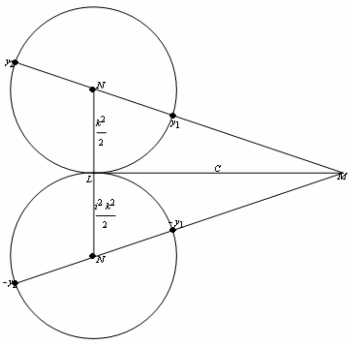
$$y=\frac{1}{2}k^2+\sqrt{\frac{1}{4}k^4-c^2}$$
$$y=\frac{1}{2}k^2-\sqrt{\frac{1}{4}k^4-c^2}$$
$$x=\pm\sqrt{\frac{1}{2}k^2+\sqrt{\frac{1}{4}k^4-c^2}} \tag{6.10}$$
$$x=\pm\sqrt{\frac{1}{2}k^2-\sqrt{\frac{1}{4}k^4-c^2}}$$

*C. Geometrical proof of the Root system*

For every real or imaginary known value *c* there are two positive or two negative y-roots and four real and four imaginary x-roots. To simplify the proofs, the order of the parts of the equation will be changed in such a way that the root factor is coming first in order. To proof the geometrical solving of a quadratic equation with the proposed Root system the René Descartes method will be used. His solution will be the starting point but instead of a/2 as the radius $k^2/2$ will be used. In these graphs the René Descartes construction will be depicted with dashed lines, Fig (6.7).

The proofs will be made for real and imaginary *k* for each of the five alternatives. For a quadratic equation with $y=x^2$ the y-axis will always be real for any combination of *k* and *x*, depending on if *k x* is real or imaginary as $k^2\,x^2$ will be real negative or positive. For complex values the absolute values will be used.

*1) Imaginary z*



$$y=\sqrt{\frac{1}{4}k^4+c^2}\pm\frac{1}{2}k^2$$
$$x=\pm\sqrt{\sqrt{\frac{1}{4}k^4+c^2}\pm\frac{1}{2}k^2}$$
$$y=-\sqrt{\frac{1}{4}k^4+c^2}\pm\frac{1}{2}k^2$$
$$x=\pm\sqrt{\sqrt{\frac{1}{4}k^4+c^2}\pm\frac{1}{2}k^2}$$

Fig. 6.6 René Descartes geometrical solution with positive $c^2$ extended for negative roots as a base for the quadratic equation in the Root system

To construct the proofs the following steps will be done:

- René Descartes solving is constructed and combined with a mirror image Fig. (6.6)
- Describe a circle with the radius equal to |c| with L as a center.
- The coefficient *k* will deform the circle to an ellipse. Construct an ellipse with the x axis |*c/k*| and y axis |c|
- Construct a positive and negative parabola $y=\pm x^2$ with vertex at origo.
- Construct hyperbolas for the ellipse.

- Draw a horizontal line through the crossing points for the parabola with the ellipse and hyperbola.
- Draw a vertical line from these crossing points to the x axis.
- Construct the x-value by using René Descartes method Fig. (6.7) for both y values.
- Make a control that the y values are equal with René Descartes method to apply the distances MO, MP with a compass from the vertical line through the crossing point between the parabola and the ellipse and hyperbola. The will result comply.

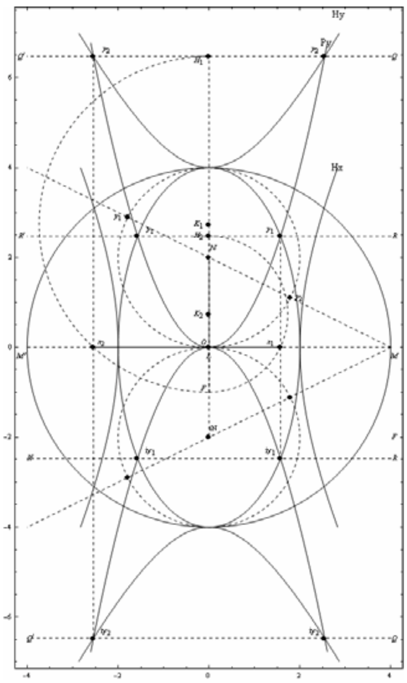
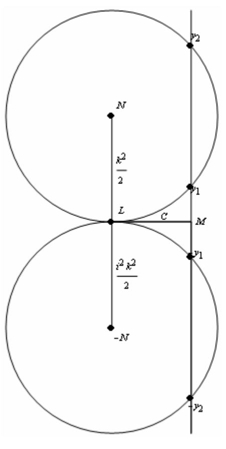


Fig. 6.7 the geometrical solution of quadratic equations in the Root system for real *z*

*2) Real z*



$$y=\sqrt{\frac{1}{4}k^4-c^2}\pm\frac{1}{2}k^2$$

$$x\pm\sqrt{\sqrt{\frac{1}{4}k^4-c^2}\pm\frac{1}{2}k^2}$$

$$y=-\sqrt{\frac{1}{4}k^4-c^2}\pm\frac{1}{2}k^2$$

$$x=\pm\sqrt{\sqrt{\frac{1}{4}k^4-c^2}\pm\frac{1}{2}k^2}$$

Fig. 6.8 René Descartes geometrical solution with negative $c^2$ and extended for negative roots as a base for the quadratic equation in the Root system.

To construct the proofs the following steps will be done:

- René Descartes solving is constructed with radius $|\frac{k^2}{2}|$ Fig (6.2) and combined with a mirror image Fig (6.8)
- The proof will be completed in the same way as previous proof.

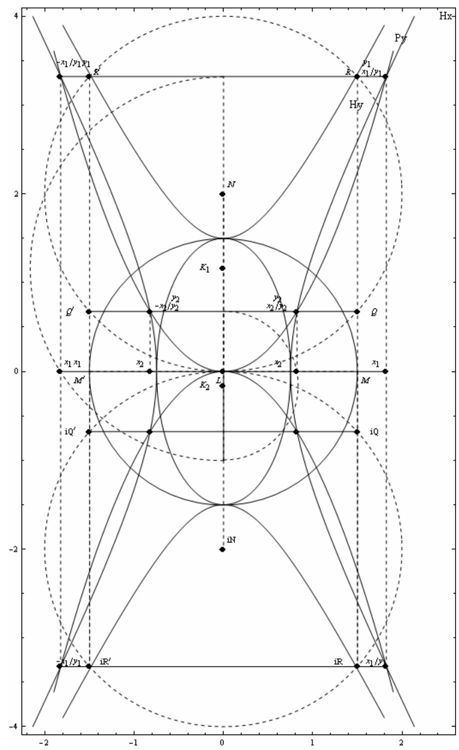
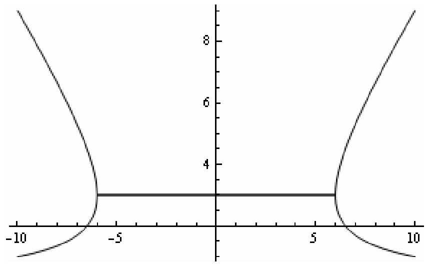


Fig. 6.9 the geometrical solution of quadratic equations in the Root system for imaginary *z*

*3) Complex roots*

If the value of the square root is imaginary, the result will be a complex number. To get a complex number the expression under the root sign will be divided by *-1* and then which gives the squared root multiplied by *i* and the result will be the complex numbers (6.15). The absolute value of the expression is constant for *-2|c|<k <2c|* and will be equal to |*z*|, Fig. (6.10)



$$y=i\sqrt{c^2-\frac{1}{4}k^4}\pm\frac{1}{2}k^2$$

$$y=-i\sqrt{c^2-\frac{1}{4}k^4}\pm\frac{1}{2}k^2$$

Fig. 6.10 The absolute value of the expression for *-2|c|<k <2<|c|*

To construct the proofs the following steps will be done:

- René Descartes geometrical solution is constructed with radius $|\frac{k^2}{2}|$ Fig (6.2) and combined with a mirror image Fig.(6.8)
- Describe a circle with the radius equal to |c| with L as a center.
- The coefficient *k* will deform the circle to an ellipse. Construct an ellipse with the x axis $|c/k|$ and *y* axis $|c|$
- Construct a positive and negative parabola $y = \pm x^2$ with vertex at origo.
- Construct hyperbolas for the ellipse.
- Draw a horizontal line as a tangent to the circle crossing the parabolas that is the $|y|$- value Fig. (6.10)Draw a vertical line from the crossing down to the x-axis, that is $|x|$ -value
- Construct another graph with the squared values of the coordinates.
- The result is in agreement with $y^2 + k^2x^2 + z^2 = 0$ and y axis |c|

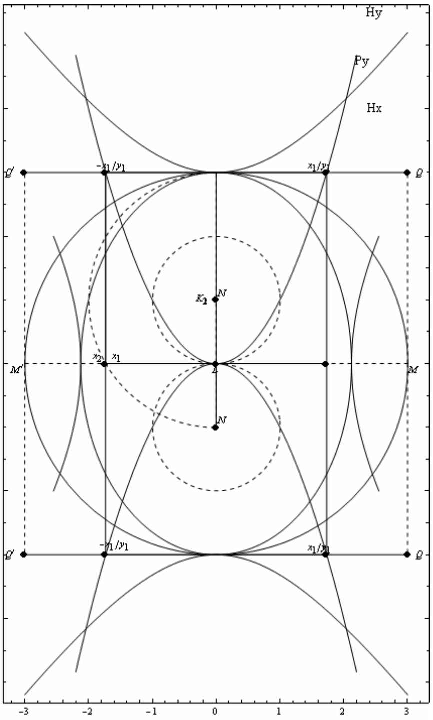



Fig. 6.11 the geometrical solution of quadratic equations in the Root system for imaginary *z*

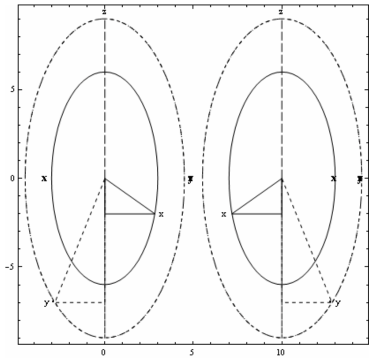


Fig. 6.12 Expanded complex roots

*4) Complex roots and complex coefficient k and c*

For the construction of the roots, René Descartes method will be partly used Fig (6.2).

- René Descartes method is constructed with radius $|\frac{k^2}{2}|$ Fig(6.2)
- Describe a circle with the radius equal to |c| with L as a center.
- The coefficient *k* will deform the circle to an ellipse. Construct an ellipse with the x axis $|c/k|$ and y axis |c|
- Construct a positive and negative parabola $y = \pm x^2$ with vertex at origo.
- Construct hyperbolas for the ellipse.
- Draw a horizontal line with the calculated |y|- values crossing the parabolas.
- Draw a vertical line from the crossing down to the x-axis, that is |x| -value
- Construct another graph with the squared values of the coordinates. The result is in agreement with (3.8). $y^2 + k^2x^2 + z^2 = 0$

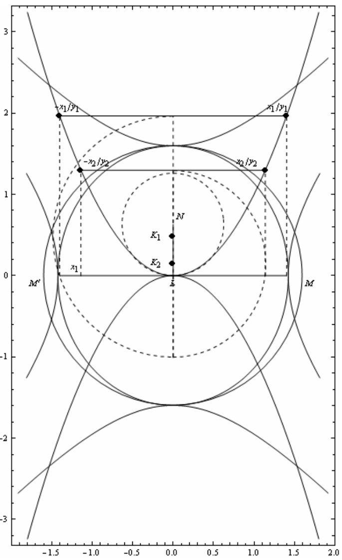


Fig. 6.13 the geometrical solution of quadratic equations in the Root system for imaginary *z*

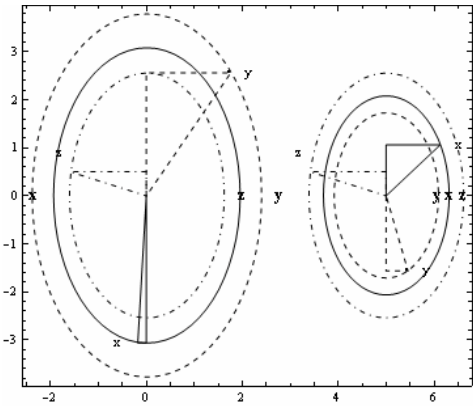


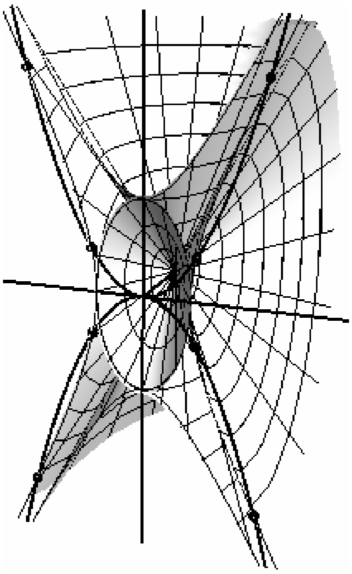
Fig. 6.14 Expanded complex roots

## D. *Some examples of solved equations with the Root system*

In this chapter an equation of each of the five different types of equations will be solved and the result plotted.

### 1) *Example 6A Real window*

The following equation with *k=2, 2i* and *z=1.5* is solved and the result is shown for both real and imaginary *k* in the same graph Fig. 6.15.

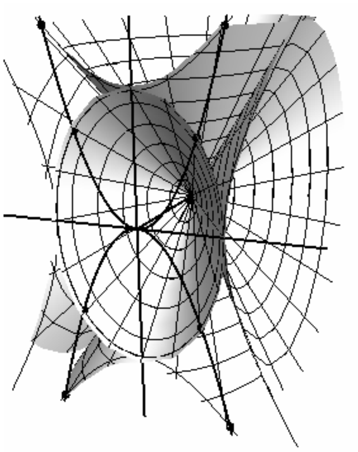


$$y = x^2$$
$$k = 2$$
$$z = 1.5$$
$$x^4 + 2^2 x^2 + 1.5^2 = 0$$
$$y_1 = -3.32288, y_2 = -0.677124$$
$$x_{1,2} = \pm 1.82288i . x_{3,4} = \pm 0.82288i$$
$$k = 2i$$
$$x^4 - 2^2 x^2 + 1.5^2 = 0$$
$$y_1 = +3.32288, y_2 = +0.677124$$
$$x_{1,2} = \pm 1.82288i . x_{3,4} = \pm 0.82288i$$

Fig. 6.15 Real window, Example 6A

### 2) *Example 6B*

The following equation with *k=1.5, 1.5i* and *z=3i* is solved and the result is shown in the same graph.

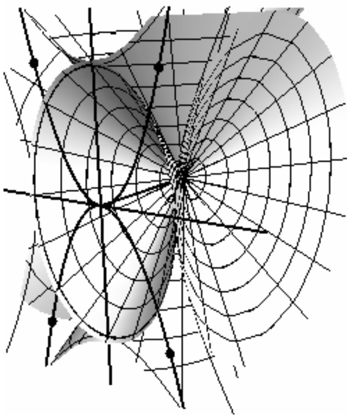


$$y = x^2$$
$$k = 1.5$$
$$z = 3i$$
$$x^4 + 1.5^2 x^2 - 3^2 = 0$$
$$y_1 = -4.329, y_2 = +2.079$$
$$x_{1,2} = \pm 2.08063i, x_{3,4} = \pm 1.44187$$
$$k = 1.5i$$
$$x^4 - 1.5^2 x^2 - 3^2 = 0$$
$$y_1 = +4.329, y_2 = -2.079$$
$$x_{1,2} = \pm 2.08063 . x_{3,4} = \pm 1.44187i$$

Fig. 6.16 Imaginary window, Example 6B

### 3) *Example 6C*

*The following equation with k=1.5, 1.5i and z=3 is solved and the result is shown in the same graph.*



$$y = x^2$$
$$k = 1.5$$
$$z = 3$$
$$x^4 + 1.5^2 x^2 + 3^2 = 0$$
$$y_1 = -1.125 - 2.78107i$$
$$y_2 = -1.125 + 2.78107i$$
$$x_{1,2} = -0.968246 \pm 1.43614i$$
$$x_{3,4} = 0.968246 \pm 1.43614i$$
$$k = 1.5i$$
$$x^4 - 1.5^2 x^2 + 3^2 = 0$$
$$y_1 = +1.125 - 2.78187i$$
$$y_2 = +1.125 + 2.78187i$$
$$x_{1,2} = +1.43614 \pm 0.968246i$$
$$x_{3,4} = -1.43614 \pm 0.968246i$$

Fig. 6.17 Real window, Example 6C

The result in Fig. 6.17 is the absolute value for the axes (3.7). To get the actual values for the variables the results has to be squared (3.8). The result is showed in Fig. 6.18

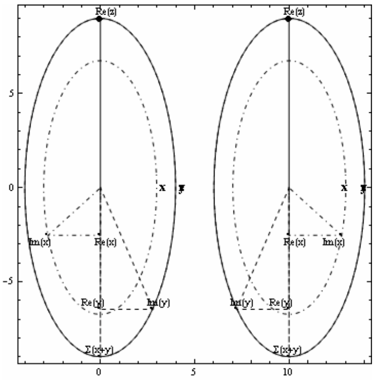
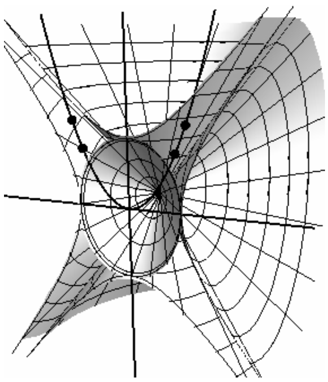


Fig. 6.18 Expanded complex roots, Example 6C

#### 4) *Example 6E*

The following equation with $k = \sqrt{\frac{2+i}{1+i}}$ and $z = \sqrt{\frac{3-2i}{1+i}}$ is solved and the result is shown in the same graph.



$y = x^2$

$z = \sqrt{\frac{3-2i}{1+i}}$

$k = \sqrt{\frac{2+i}{1+i}}$

$y^2 + k^2x^2 + z^2 = 0$

$y_1 = -1.78078 - 0.780776i$

$y_2 = +0.280776 + 1.28078i$

$x_{1,2} = \mp 0.286047 \pm 1.36477$

$x_{3,4} = \mp 0.892179 \pm 0.71778i$

Fig. 6.19 Complex window, Example 6E

The results in Fig. 6.19 are the absolute value for the axes (3.7). To get the actual values for the variables the results has to be squared (3.8). The result is showed in Fig. 6.20

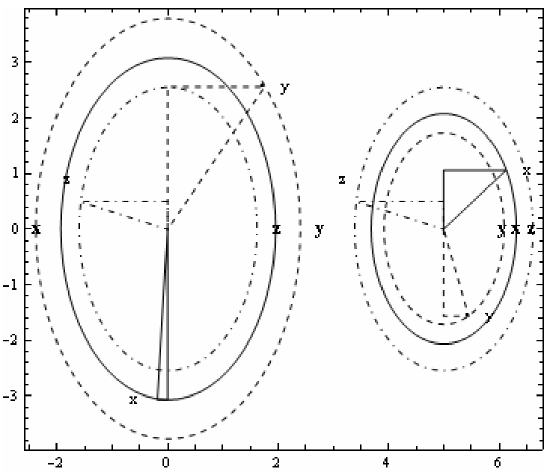


Fig. 6.20 Expanded complex roots, Example 6E

## VII. Cubic Equations

### A. *The Algebra of Omar Khayyam [3]*

Omar Kayyam has in his work used the same method as used in the Root system but he has not expressed it geometrical. He is using in many places in his work expressions like $BE^2 = AB \cdot ED$ which correspond to the expressions derived and used in this work.)

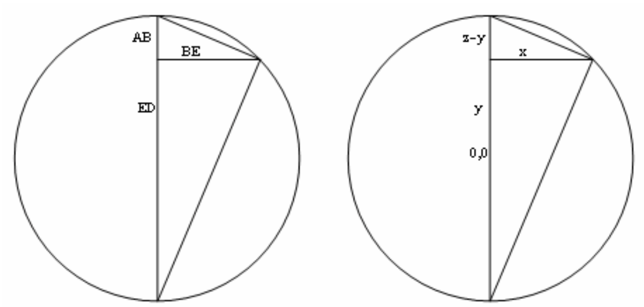



Fig. 7.1 comparison between Omar Khayyam's geometry and the Root system for cubic equation with real coefficient *k*

His proof for what he calls the first species will be recited.

#### 1) *A cube and sides are equal to a number*

$$x^3 + bx = a \tag{7.1}$$

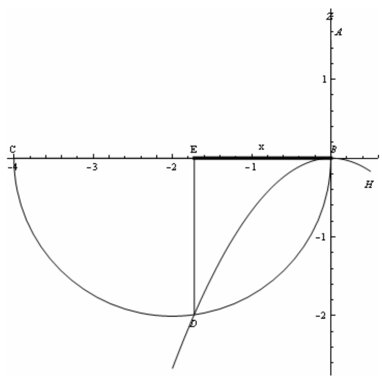



Fig. 7.2 Omar Khayyam's first species

Let the line AB Fig. (7.2) be the side of a square equal to the given number of roots[1]. Construct a solid whose base is equal to the square on AB, equal in volume to the given number. The construction has been shown previously[2]. Let BC be the height of the sold. Let BC be perpendicular to AB. You know already what meaning is applied in this discussion to the phrase solid number. It is a solid whose base is the square of unity and whose height is equal to the given number; that is the height is a line whose ratio to the side of the base of the solid is as the ratio of the given number to one. Produce AB to Z and construct a parabola whose vertex is the point B, axis BZ, and parameter AB. Then the position of the conic HBD will be known, as has been shown previously as it will be tangent to BC. Describe on BC a semicircle. It necessarily intersects the conic. Let the point of intersection be D; drop from D, whose position is known, two perpendiculars DZ and DE on BZ and BC[3]. Both the position the magnitude of these lines is known. The line DZ is an ordinate of the conic. Its square is the equal to the product of BZ and AB.[5] Consequently, AB to DZ which is equal to BE, is as BE to ED, which is equal to ZB[4]. But BE to ED is as ED to EC.[5] The four lines then are in continuous proportion, AB, BE, ED, EC,[6] and consequently the square of the parameter AB, the first, is to the square of BE, the second, as BE, the second, is to EC, the fourth[7]. The solid whose base is the square AB and whose height is EC is equal to the cube BE, because the heights of these figures are reciprocally equal to there bases[8]. Let the solid whose base is the square of AB and height is EB be added to both[9]. The cube BE plus the solid then is equal to the solid whose base is the square AB and whose height is BC, which solid we have assumed to be equal to the given number. But the solid whose base is the square of AB, which is equal to the number of roots, and whose height is EB, which is the side of the cube, is equal to the number of its given sides is equal to the given number, which was required.

1. . $AB=\sqrt{b}, \rightarrow \overline{AB}^2=b$
2. $\overline{AB^2}\cdot BC=a, \rightarrow BC=\frac{a}{b}$
3. $\overline{DZ}^2=AB\cdot BZ \rightarrow AB:DZ=DZ:BZ$
4. $DZ=EB, BZ=ED \rightarrow AB:BE=BE:ED$
5. *DE, a perpendicular drawn from a point on the circumference to the diameter, is the mean proportional between the segment BE and ED* $(BE:ED=ED:EC)$
6. *That is, AB:BE=BE:ED=ED:EC*
7. *Multiplying the term of propotion AB:BE=BE:ED by the corresponding terms of AB:BE=ED:EC, we obtain* $\overline{AB^2}\cdot \overline{EC^2}=\overline{BE^2}$.
8. The last proportion gives $\overline{AB}^2\cdot EC=\overline{BE}^3$
9. If, $\overline{AB}^2\cdot EC=\overline{BE}^3, \overline{AB}^2\cdot EC+\overline{AB}^2\cdot EB=\overline{BE}^3+\overline{AB}^2\cdot EB.$ ,or, But BC=a/b and $\overline{AB}^2=b.$ Therefore $BE+bBE=b\frac{a}{b}=a$ ,Hence BE satisfies the equation $x^3+bx=a$

## B. The Root system

Omar Kayyam has in his proofs used lines, circle, parabola and hyperbola. In this work the cube function $y=x^3$ will be used instead of the parabola.

For the geometrical construction the same presumption will be used as for the previous equations. If we use, $y=x^3$ , and apply this in Fig.(3.5), (3.6). We will get a reduced cubic equation (7.2), Fig. (7.3):

$$\begin{aligned}&\text{For real } z\\ &x^6+k^2x^2+z^2=0\\ &\text{For imaginary z}\\ &x^6+k^2x^2-z^2=0\end{aligned} \tag{7.2}$$

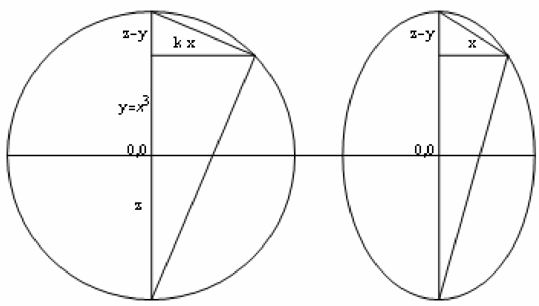



Fig. 7.3 the base for the Root system for a cubic equation with real coefficient *k*

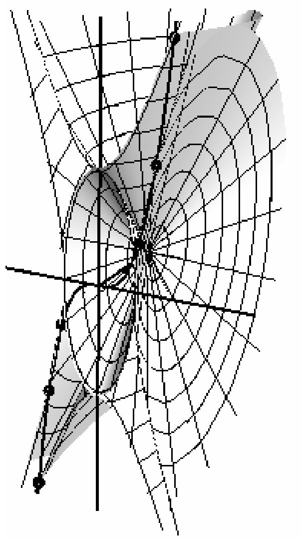
## C. Some examples of solved cubic equations with the Root system

In this part an equation of each of the three different types of equations will be solved and the result depicted.

| | |
|---|---|
| $\frac{z4}{4}+\frac{k^4}{27}>0$ | There will be one real root and two imaginary roots |
| $\frac{z4}{4}+\frac{k^4}{27}=0$ | There will be three real roots of which at least two are equal |
| $\frac{z4}{4}+\frac{k^4}{27}<0$ | There will be three real and unequal roots |

### 1) Example 7A

The following equation with *k=3i* and *z=3i* is solved. The result will be one real root and two imaginary roots on the x-axis, and y-axis. The result is shown in the graph. 7.4)



$y=x^3$

$x^6-3^2x^2-3^2=0$

$x_{1,2}=\pm 1.84702,$

$y_{1,2}=\mp 6.30105$

$x_{3,4}=\pm 1.49221i,$

$y_{3,4}=\mp 3.32267i$

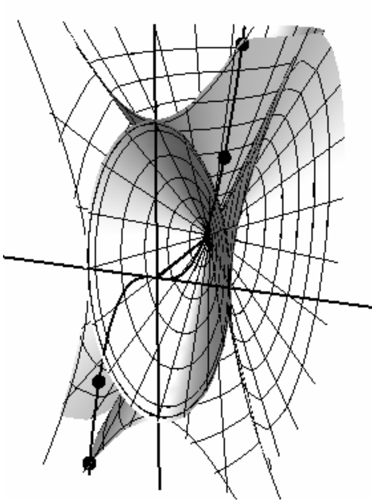
$x_{5,6}=\pm 1.08848i$

$y_{5,6}=\mp 1.28963i$

Fig.7.4 Real window, Example 7A

### 2) Example 7B

The following equation with *k=3i* and *z=2i* is solved. There will be one real root and two imaginary roots on the x-axis, The result is shown in the graph. 7.5)



$$y = x^3$$
$$x^6 - 3^2x^2 - 2^2 = 0$$
$$x_{1,2} = \pm 1.64515$$
$$y_{1,2} = \pm 4.45265$$
$$x_{3,4,5,6} = \pm 0.48491 \pm 1.26032i,$$
$$y_{3,4,5,6} = \pm 2.19667 \pm 1.11285i$$

Fig. 7.5 Imaginary window, Example 7B

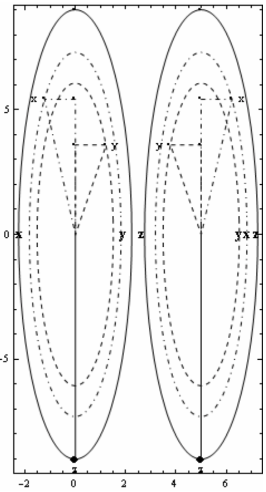
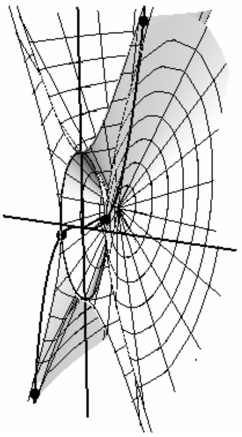


Fig. 7.6 Expanded complex roots, Example 7B

3) *Example 7C*

The following equation with *k=3* and *z=2i* is solved there will be one real root and two imaginary roots on the x-axis, The result is shown in the Fig.7.7)



$$y = x^3$$
$$x^6 + 3^2x^2 - 2^2 = 0$$
$$x_{1,2} = 1.18612 \pm 1.27457i,$$
$$y_{1,2} = -4.11192 \pm 3.30895i$$
$$x_{3,4} = -1.18612 \pm 1.27457i$$
$$y_{3,4} = 4.11192 \pm 3.30895i$$
$$x_{5,6} = \pm 0.659758$$
$$y_{5,6} = \pm 0.28718$$

Fig. 7.7 Imaginary window, Example 7C

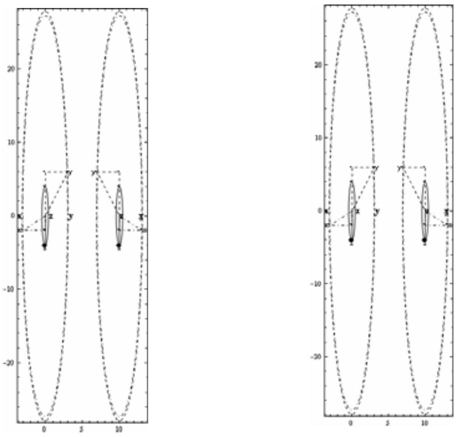
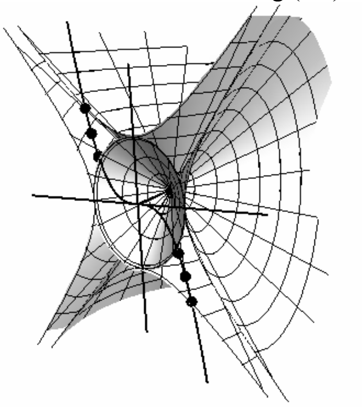


Fig. 7.8 Expanded complex roots, Example 7C

4) *Example 7D*

The following equation with $k = \sqrt{\frac{2+i}{1+i}}$ $z = \sqrt{\frac{3-2i}{1+i}}$ is solved and the result is shown in Fig (7.9).



$$y = x^3$$
$$x^6 + \frac{2+i}{1+i}x^2 + \frac{3-2i}{1+i} = 0$$
$$x_{1,2} = \pm 0.913529 \mp 0.939874i$$
$$y_{1,2} = \mp 1.65856 \pm 1.52282i$$
$$x_{3,4} = \mp 0.329433 \pm 0.976553i,$$
$$y_{3,4} = \pm 0.906745 \mp 0.613352i$$
$$x_{5,6} = \pm 1.07032 \mp 0.50162i$$
$$y_{5,6} = \pm 0.418189 \pm 1.59772i$$

Fig. 7.9 Complex window, Example 7D

The result in Fig. 7.9 is the absolute value for the axes (3.7). To get the actual values for the variables the results has to be squared (3.8). The result for the roots are shown in Fig. 7.10-7.12

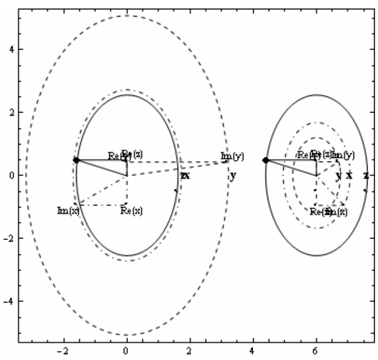


Fig. 7.10 Expanded complex roots, Example 7D

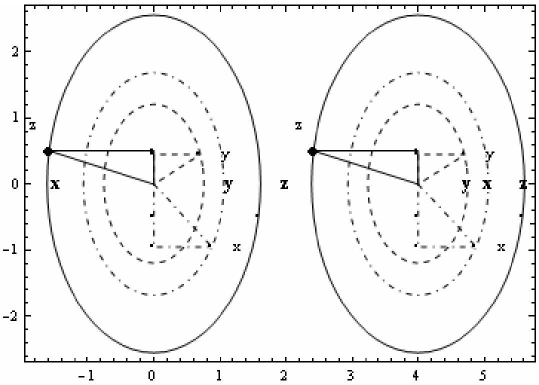


Fig. 7.11 Expanded complex roots, Example 7D

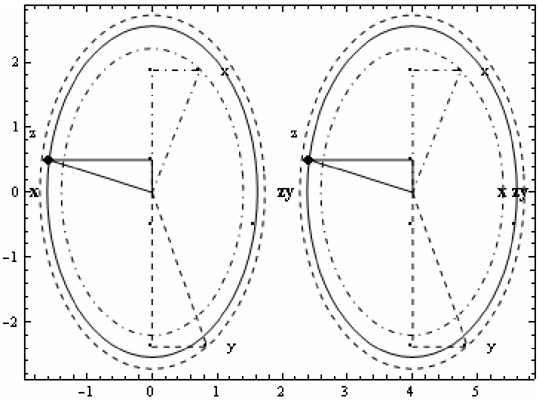


Fig. 7.12 Expanded complex roots, Example 7D

## VIII. QUARTIC EQUATIONS

### *A. The Root system*

For the geometrical construction the same presumption will be used as for the previous equations. If we use, $y = x^4 + c^2$ and apply this in Fig (3.5, 3.6) we will get a reduced quartic equation (8.1),
Fig (7.3):

$$x^8 + 2c^2x^4 + k^2x^2 + z^2 + c^4 = 0 \quad (8.1)$$

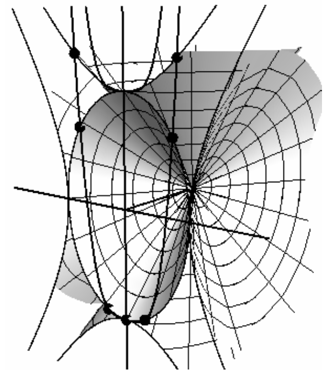
### *B. Some examples of solved quartic equations with the Root system*

The imaginary roots for quartic equations on the real side of the coordinate system falls on the quartic function (8.2)

$$y = \pm(x^4 + c^2) \quad (8.2)$$

#### *1) Example 8A*

The following equation with *k=1.5,* $c = \sqrt{3}i$ and $z = 3i$ is solved. The result is shown in Fig. (8.1)



$y = x^4 + (\sqrt{3}i)^2$

$y^2 + 1.5^2 x^2 - 3^2 = 0$

$x_{1,2} = \pm 1.61833i, y_{1,2} = 3.85911$

$x_{3,4} = \pm 0.620058, y_{1,2} = -2.85218,$

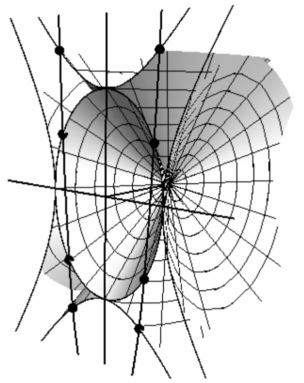
$x_{5,6} = \pm 1.49483, y_{5,6} = 1.99307$

$x_{7,8} = 0, y_{7,8} = -3.0$

Fig. 8.1 Real window, Example 8A

#### *2) Example 8B*

The following equation with *k=1.5i, c=2.5i* and *z=3i* is solved. The result will be one real root and two imaginary roots on the x-axis, The result is shown in Fig. (8.2)



$y = x^4 + (2.5i)^2$

$y^2 - 1.5^2 x^2 - 3^2 = 0$

$x_{1,2} = \pm 1.79045, y_{1,2} = 4.02652$

$x_{3,4} = \pm 1.28042, y_{3,4} = -3.56214$

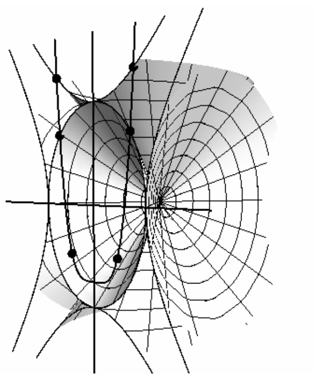
$x_{5,6} = \pm 1.42714i, y_{5,6} = -2.10175$

$x_{7,8} = \pm 1.67584i, y_{7,8} = 1.63737$

Fig. 8.2 Imaginary window, Example 8B

#### *3) Example 8C*

The following equation with k=1.5i, c=1.5i and z=3i is solved. The result will be one real root and two imaginary roots on the x-axis, Fig. (8.3)



$y = x^4 + (1.5i)^2$

$y^2 - 1.5^2 x^2 - 3 = 0$

$x^8 - 4.5x^4 - 2.25x^2 - 3^2 + 1.5^4 = 0$

$x_{1,2} = \pm 1.56919, y_{1,2} = 3.81317$

$x_{3,4} = 0.585558 \pm 0.73064i$

$y_{3,4} = -2.9457 - 0.326789i$

$x_{5,6} = -0.585558 \pm 0.73064i$

$y_{5,6} = -2.9457 - 0.326789i$

$x_{7,8} = \pm 1.44237i, y_{7,8} = 2.07822$

Fig. 8.3 Imaginary window, Example 8C

To get the actual values for the variables the results has to be squared (3.8). The result for two of the roots are shown in Fig.8.4

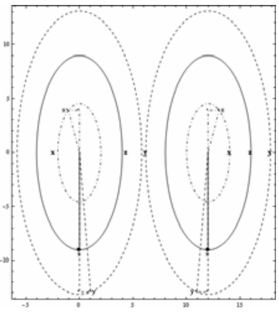
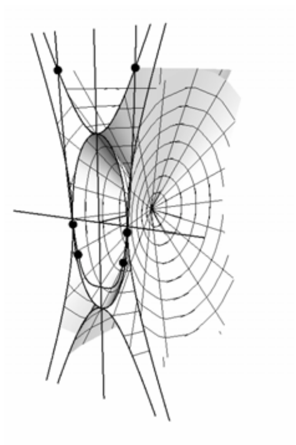


Fig. 8.4 Expanded complex roots Example 8C

### C. *Example 8D*

The following equation with *k=2.5, c=1.5i* and *z=3i* is solved. The result in will be one real root and two imaginary roots Fig (8.7)



$$y = x^4 + (1.5i)$$

$$y^2 + 2.5^2 x^2 - 3^2 = 0$$

$x_{1,2} = \pm 1.64613i, y_{1,2} = 5.09273$

$x_{3,4} = 0.906782 \pm 0.429395i$

$y_{3,4} = -2.44954 \pm 0.993468i$

$x_{5,6} = -0.906782 \pm 0.429395i$

$y_{5,6} = -2.44954 \pm 0.993468i$

$x_{7,8} = \pm 1.1975, y_{7,8} = -0.193643$

Fig. 8.5 Imaginary window, Example 8 D

To get the actual values for the variables the results has to be squared (3.8). The results for two of the roots are shown in Fig. (8.6)

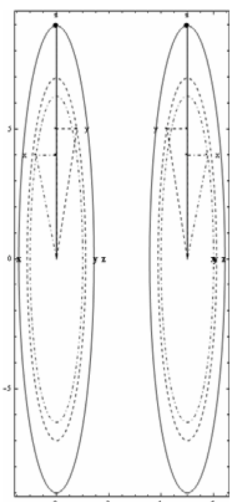
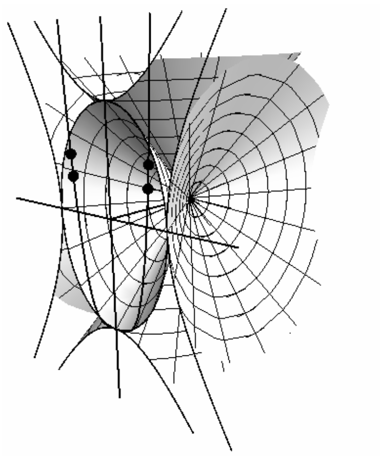


Fig. 8.6 Expanded complex roots Example 8D.

### D. *Example 8E*

The following equation with *k=1.5,* $c = \sqrt{3}i$ and *z=3* is solved. The result in Fig.(8.7) will be one real root and two imaginary roots on the x-axis,



$$y = x^4 + (\sqrt{3}i)^2$$

$$y^2 + 1.5^2 x^2 + 3^2 = 0$$

$x_{1,2} = -1.420742 \pm 0.337896i$

$y_{1,2} = -0.29538 \pm 3.65677i$

$x_{3,4} = 1.420742 \pm 0.337896i$

$y_{3,4} = -0.29538 \pm 3.65677i$

$x_{5,6} = 0.20619 + 1.39529i$

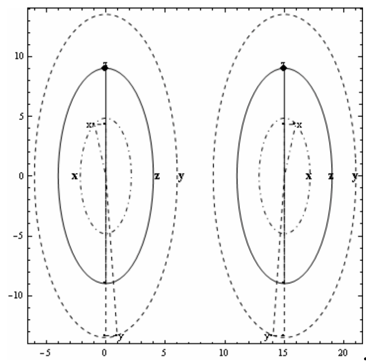
$y_{5,6} = 0.29538 \pm 2.19146i$

$x_{7,8} = -0.20619 + 1.39529i$

$y_{7,8} = 0.29538 \pm 2.19146i$

Fig. 8.7 Imaginary window , Example 8E

To get the actual values for the variables the results has to be squared (3.8). The result for the roots is shown in Fig. (8.8),(8.10)



,

Fig. 8.8 Expanded complex roots Example 8E

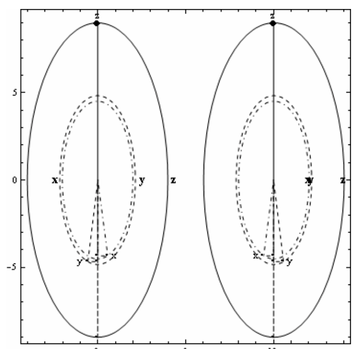


Fig. 8.9 Expanded complex roots Example 8E

### 1) *Example 8F Complex*

The following equation is solved. The result will be eight complex roots

$$y = x^4 + 2^2$$

$$y^2 + k^2 x^2 + z^2 = 0$$

$$k = \sqrt{\frac{2+i}{1+i}} \quad z = \sqrt{\frac{3-2i}{1+i}}i \quad c = 2$$

$x_1 = -1.18764 - 1.00493i$
$y_1 = -1.5373 + 1.91251i$
$x_2 = 1.18764 + 1.00493i$
$y_2 = -1.5373 + 1.91251i$
$x_3 = 0.77141 + 1.03726i$
$y_3 = 1.67021 - 1.53899i$
$x_4 = -0.77141 - 1.03726i$
$y_4 = 1.67021 - 1.53899i$
$x_5 = 1.09256 - 1.05364i$
$y_5 = 1.29376 - 0.384641i$
$x_6 = -0.916975 + 0.918772i$
$y_6 = 1.16085 + 0.0111223i$
$x_7 = -1.09256 + 1.05364i$
$y_7 = -1.29376 - 0.384641i$
$x_8 = 0.916975 - 0.918772i$
$y_8 = 1.16085 + 0.0111223i$

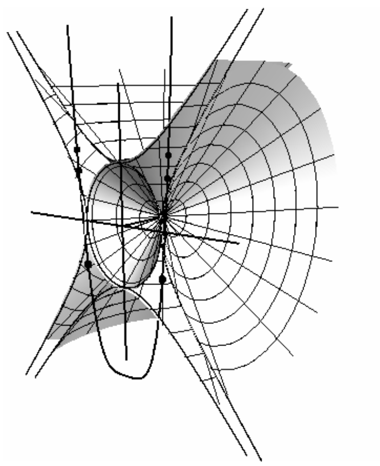


Fig. 8.10 Complex window, Example 8F

To get the actual values for the variables the results has to be squared (3.8). The result for two of the roots in each figure are shown in Fig.8.11-8.14

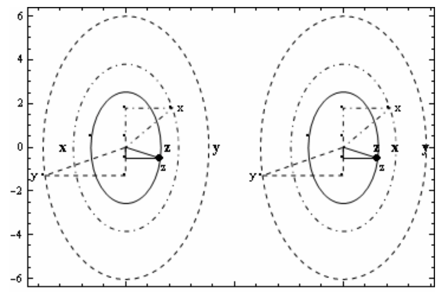


Fig. 8.11 Expanded complex roots Example 8F

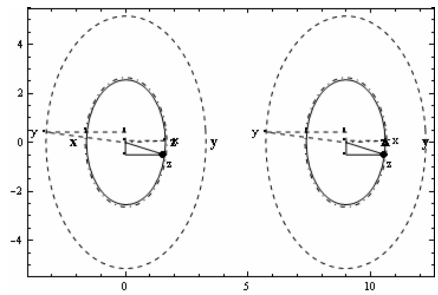


Fig. 8.12 Expanded complex roots Example 8F

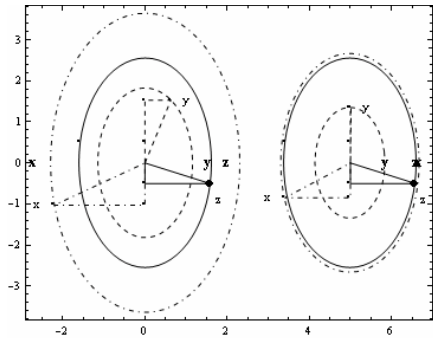


Fig. 8.13 Expanded complex roots Example 8F

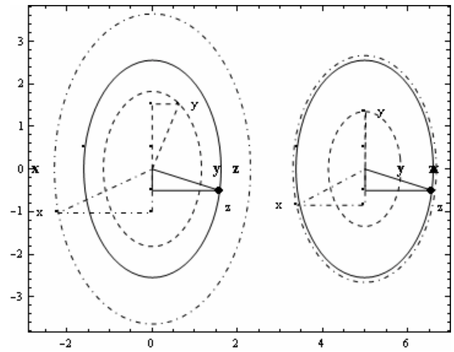


Fig. 8.14 Expanded complex roots Example 8F

### 2) *Example 8G*

If the coefficient $a = 0$ the absolute values of the roots will be limited between the ellipse and the hyperbola

$$y = x^4$$

$$y^2 + \left(\sqrt{\frac{2+i}{1+i}}\right)^2 x^2 - \left(\sqrt{\frac{3-2i}{1+i}}\right)^2 = 0$$

$x_1 = -0.149384 - 1.19631i$
$y_1 = 1.8571 \; - 1.0071\, i$
$x_2 = 1.23481 \; - 1.0123\, i$
$y_2 = 1.01772 \; + 0.633102\, i$
$x_3 = -1.25745 + 1.63644\, ä,i$
$y_3 = 1.23481 \; - 1.0123\, i$
$x_4 = -1.01772 - 0.633102\, i$
$y_4 = -1.25745 + 1.63644\, i$
$x_5 = 1.23481 \; - 1.0123\, i$
$y_5 = -1.8571 \; - 1.0071 \; i$
$x_6 = -0.149384 \; - 1.19631\, i$
$y_6 = 1.8571 - 1.0071i$
$x_7 = -1.23481\text{-}1.0123i$
$y_7 = 0.607369 - 1.00949\, i$
$x_8 = 1.23481 - 1.0123i$
$y_8 = -1.00754 + 0.265105\, i$

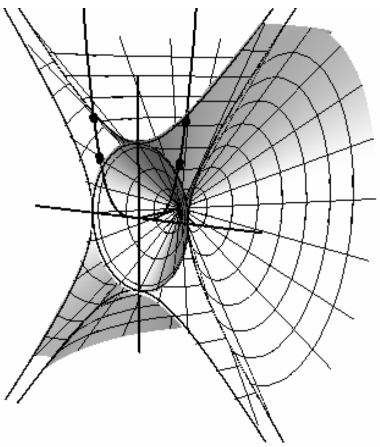


Fig. 8.15 Complex window, Example 8G

To get the actual values for the variables the results has to be squared (3.8). The result for two of the roots in each figure are shown in Fig.8.16-8.19

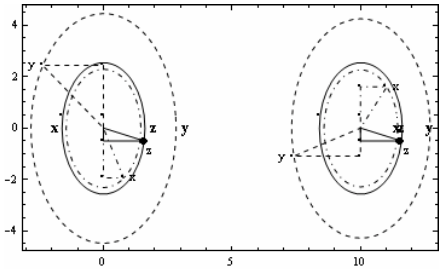


Fig. 8.16 Expanded complex roots Example 8G

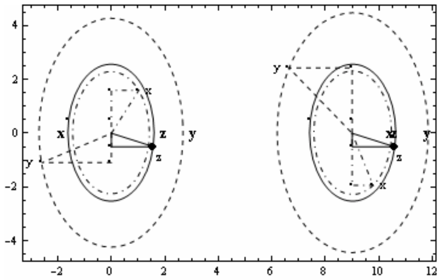


Fig. 8.17 Expanded complex roots Example 8G

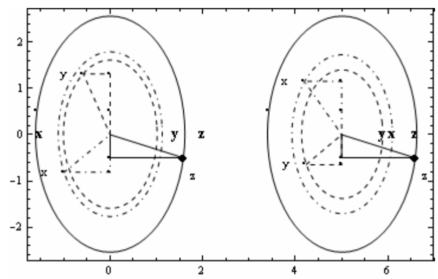


Fig. 8.18 Expanded complex roots Example 8G

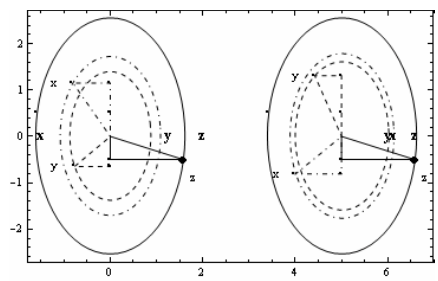


Fig. 8.19 Expanded complex roots Example 8G

## IX. Conclusion

The Root system is a local system attached to the object and function as a communicator to the surrounding space. The system adjusts it self depending on the surrounding conditions. An object travelling with a speed near to the speed of light the space for the object will change as in example 9F. The Root system seems to have many applications in physics and further research in the field is interesting. The equations have special characteristics as the solutions to the equations are dualistic. They can be solved both algebraic and trigonometric as shown in the example (5A, 5B). The solutions of the equations have there roots limited between the circle/ellipse and the hyperbola. This means for the linear equation if the angle for the vector is $\varphi = 0 + n \cdot \pi / 2$ there are no complex roots. If the vector is $\varphi = \pi / 4 + n \cdot \pi / 2$ there are complex roots which can take absolute values between the ellipse and infinity.

### A. *Wave function*

The linear equation can make wave functions with the coefficients $\alpha$ as a time function together with the coefficient *k* to determine the wave length and *z* the amplitude.

### B. *Hilbert Space*

Hilbert space is an inner product space $HxH \to \square$ , actually a root system to depict complex roots. Hilbert space has a complex plane associated with it. The linear equations 5E together with Fig 5.5 and Fig 5.6 must then be a result of Hilbert space, so even equation 5J and 5K. Even the result from the complex roots from quadratic cubic and quartic equations must be a result in agreement with Hilbert space

### C. *Dirac equation*

Dirac's equation is expressed as:

$$\frac{E^2}{c^2} - p^2 = m^2 c^2$$

If the equation is reformulated as:

$$c^4 + \frac{p^2}{m^2} c^2 - \frac{E^2}{m^2} = 0$$

With $k = \frac{p}{m}$ and $z = \frac{E}{m}$ the equation becomes a quadratic equation of the Root system with the speed of light as the variable. As the speed of light is constant then the *k* coefficient will change if *z* is known or vice versa, which means that the ellipse and corresponding hyperbolas will change the shape of the circle accordingly.

### D. *Many-Wolds interpretation MWI*

Many–World's interpretation, parallel universes etc. is an interpretation of quantum mechanics. The parallel world is even the limit of the Root system. When the coefficient $k \to 0$ the y-axis will contract to the level of the *z* value and a parallel world occur. If the axis of the windows is extended from $\pi/2$ to $n\pi$ a number of parallel universes will occur. The distance between the universes will be equal to the value of *z*.

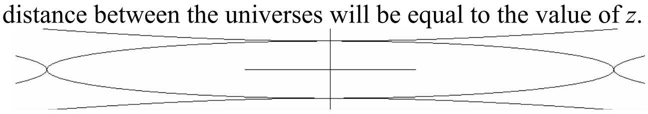
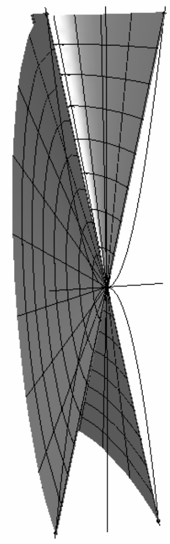


Fig. 9.1 Many–World's interpretation

If $k \to \infty$ the window of the x-axis cone will unite with the compressed y-axis cone and the system will be 2-dimentional Fig 9.1

### E. Lorentz transformation

If we use the Root system to depict the Lorentz transform for a speed of 99.9% of the light with the following parameters:



$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-0.999^2}}$$

The following equation will depict the space:

$$x^4 + (\sqrt{\gamma}x)^2 + 1 = 0$$

The contraction is *22.33* fold

Fig. 9.2 Lorentz transformation

### F. Polarization

If the coefficient $k$ for a linear equation takes a value different from 1 the circle of the $z$ axis will change to an ellipse and when the vector rotates a polarized $x$- and $y$-wave occur.

### G. Gravitational interaction

All the heavenly bodies travel on an of an elliptic, hyperbolic or parabolic curve which suggests that the equations for there orbits can be treated and depicted by the Root system as the Root system for the quadratic equation is based on all these three curves.

### H. Particle charge

In [9] there is an example, Fig (9.3) to obtain the distance of the closest approach of a particle of charge $ve$ directed with velocity $v_0$ against an atom whose atomic number is Z. The solution of this example shows a direct compliance with the Root system.

If we in the above example use $z = \frac{e}{v}\sqrt{\frac{vZ}{4\pi\varepsilon_0 m}}$

R will be the focal point for the hyperbola on the y-axis in Fig (9.2) and if $k \to \infty$, the focal point R of the hyperbola will coincide with the radius $z$ of the equation:

$$x^4 + (ik)^2 x^2 + z^2 = 0$$

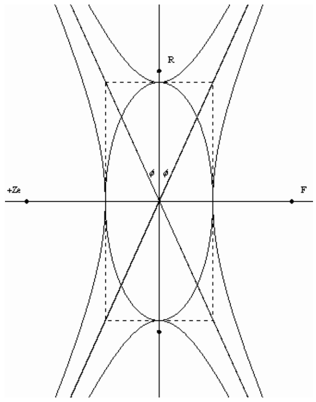
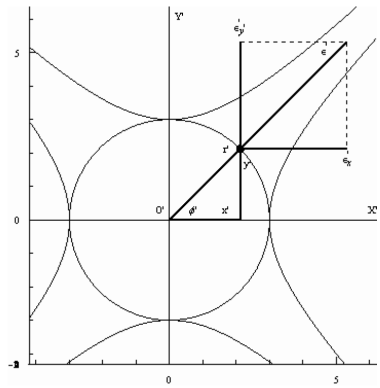



Fig. 9.3 the transformation for a charge $q$ at rest

### İ. Electromagnetic field of a moving charge

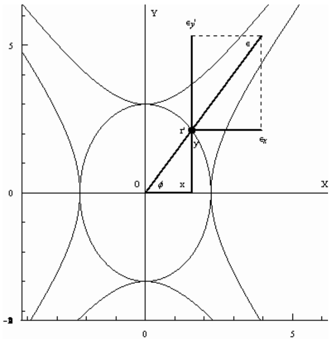
In [9] is a derivation of how the electric field for a moving charge is changing because of the relativistic transformation of the electromagnetic fields. The field is deformed by the movement of a factor $\sqrt{1-v^2/c^2}$ and the linear equation from the Root system can be used to depict the result. Fig 9.3 depicts the transformation for a charge $q$ at rest. Fig. 9.4 depict the transformation of a moving charge $q$




$$y = \alpha x$$
$$z = r'$$
$$k = \sqrt{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}}$$
$$y^2 + k^2x^2 + z^2 = 0$$
$$r = \sqrt{x^2 + y^2}$$

Fig. 9.4 Relativistic transformation of the components of the electric field produced by a moving charge $q$ relative to O and located at O'

$$r' = z$$
$$y = \alpha x$$
$$\alpha^2x^2 + k^2x^2 - z^2 = 0$$
$$x^2 = z^2\frac{1}{\alpha^2 + k^2}$$
$$y^2 = z^2\frac{\alpha^2}{\alpha^2 + k^2}$$
$$r^2 = z^2\left(\frac{1}{\alpha^2 + k^2} + \frac{\alpha^2}{\alpha^2 + k^2}\right) = z^2\frac{1+\alpha^2}{\alpha^2 + k^2}$$
$$\varepsilon = \frac{q}{4\pi\varepsilon_0 r^3}\left[\frac{(\alpha^2 + k^2)}{1+\alpha^2}\right]^{\frac{3}{2}} r = \frac{q}{4\pi\varepsilon_0 r^2}\left[\frac{(\alpha^2 + k^2)}{1+\alpha^2}\right]^{\frac{3}{2}} u_r$$



$$y = k\alpha x$$

$$k = \sqrt{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}}$$

$$z = r'$$

$$y^2 + k^2x^2 + z^2 = 0$$

$$r = \sqrt{x^2 + y^2}$$

Fig. 9.5 Relativistic transformation of the components of the electric field produced by a moving charge *q* relative to O and located at O

### *J. Light*

Light is an electromagnetic wave. The linear equation of the Root system is tree dimensional with the *x*- and *y*- axes perpendicular to each other. If the cosines function of the *x*-axis represent the magnetic field and the sinus function of the *y*-axis the electric field the outcome when the radius function rotate a wave function similar to a light wave will occur. The wave length, amplitude and the polarization is depending on the coefficient *k* and the value of *z*.

### *K. Quark*

Finally it is very tempting to give the six cones each one a name like Up, Down, Charm, Strange, Top, Bottom, due to the experience received when solving quadratic equations. There are five different types of quadratic equation where the solution engages the different axis in different ways, depending on the coefficient *k* and the diameter *z*. With the coefficient *k* as a time machine and real and imaginary *z* can make a mathematical model binding the different cones together in different combinations. What seem especially interesting are that the complex roots has an absolute value which is constant and equal to |z| for all complex roots except for equations with complex coefficient *k* and *z*.

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