A Role of Fourier Transform with Unitary Group

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***Abstract*—**The aim of this paper is to discuss a role of Fourier transform of sample functions of complex white noise which are generalized functions. Functionals of complex Brownian motion and the analysis of such functionals is not only useful in the study of stochastic processes, but is also widely used in applications .In this paper the infinite dimensional unitary operators on the Hilbert space $L_c^2(R^1)$ is discussed. This group gives not only group-theoretical interest but also an intrinsic view point of the complex white noise. A flow { g^*_t } from a one-parameter subgroup { g_t } of $U(E_c)$ which is obtained on the measure space. While $U(E_c)$ characterizes, in some sense, the structure of the measure ν, these observations lead to a probabilistic approach to the investigations of the infinite dimensional unitary group $U(E_c)$. The aim of this paper is to find one-parameter subgroups of $U(E_c)$ which have some connection with probability theory in particular with Brownian motion. In fact, six one-parameter subgroups which form a Lie group. The infinitesimal generators of the one-parameter subgroups are introduced and discussed their commutation relations.

***Keywords*—**Fourier transform, Hilbert space, Lie group, Schwartz space.

I. INTRODUCTION

IN this paper, the Fourier transform of sample function of the complex white noise is discussed as a basic tool. In the investigation of the sample function properties of the complex Brownian motion, the Fourier analysis plays a dominant role to discuss generalized harmonic analysis or frequency problems.

Firstly, start with general set-up of the complex white noise, let E be a σ-Hilbert nuclear space which is included densely in L^2 (R^1) and let E^* be the dual space of E. Let E_c be the complexification of E. The complex white noise gives a probability measure ν which is Gaussian, on the space E_c^*, the conjugate space of E_c. The infinite dimensional unitary group $U(E_c)$ is a group of unitary operators on L_c^2 (R^1) which leave E_c invariant. Alternatively, U (E_c) is the collection of all the linear transformations on E_c leaving the L^2-norm invariant. Here, it should be noted that the basic space E_c must be chosen so that the Fourier transform is a linear isomorphism of the space E_c ; namely the Fourier transform is a member of $U(E_c)$. The space E_c can be topologized by using the differential operator D which is invariant under the Fourier transform.

Take $E = S$ the Schwartz space of C^∞- functions which are rapidly decreasing at infinity so that the ordinary Fourier transform is member of U (E_c).

In section II, the intimate connection arises between the unitary group $U(S_c)$ and the measure ν of the complex white noise. It is proved that each g^* the conjugate of $g \in U(S_c)$ acts on S_c^* and ν is invariant under the action g^*.

A shift forms a one-parameter subgroup of $U(S_c)$. Another one-parameter subgroup can be obtained by observing the Fourier transform of fractional order which has been developed by [9]. The finite dimensional Lie subgroup of $U(S_c)$ which contains the above one-parameter subgroup and has nice algebraic properties. Such a subgroup exists as is prescribed in section III.

II. COMPLEX WHITE NOISE AND UNITARY GROUP $U(E_C)$

The following theory is well known:

Given a functional $C_\sigma(\xi) = \exp\left[-\frac{\sigma^2}{2}\|\xi\|^2\right], \xi \in E,$ and the unique probability measure μ_σ is obtained on E^* such that

$$C_\sigma(\xi) = \int_{E^*} e^{i<x,\,\xi>}\, d\mu_\sigma(x), \qquad (1)$$

where $<x, \xi>$, $x \in E^*$, $\xi \in E$, is the canonical bilinear form and $\|\ \|$ denotes the $L^2(R^1)$-norm. Thus μ_σ is called the measure of (real, Gaussian) white noise with variance σ^2.

Here, E and E^* are complexified in a usual manner as follows:

$$E_C = E + iE,\ E_C^* = E^* + iE^*$$

Obviously E_c is a vector subspace of the complex Hilbert space $L_c^2(R^1)$, and $\zeta = \xi + i\eta \in E_c$ $(\xi, \eta \in E)$ and $z = x + iy \in E_c^*$ (x, y $\in E^*$) are linked by the following canonical form

$$<z, \zeta> = (<x, \xi> + <y, \eta>) + i\,(- <x,\eta> + <y,\xi>). \qquad (2)$$

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To complexify the white noise, the product measure $\nu = \mu_{\sqrt{1/2}} \times \mu_{\sqrt{1/2}}$ is introduced on the space E_C^*. The measure space $(E_C^*, \mathcal{B}, \nu)$ or simply denoted by (E_C^*, ν) is called the complex white noise, where $\mathcal{B}$ is the σ-field generated by all cylinder subsets of E_C^*. The systems {<z, ζ>; ζ ∈ E_c} and {<z, ζ>; ζ ∈ E_c} of random variables on the space (E_C^*, ν) are both complex Gaussian system.

The infinite dimensional unitary group can be defined as follows. Consider the collection $U(E_c)$ of all linear transformations on E_c satisfying the following two conditions:

(i) g is a homeomorphism of E_c,

(ii) $\|g\zeta\| = \|\zeta\|$ for every $\zeta \in E_c$.

It can easily be proved that $U(E_c)$ become a group under the product

$$(g_1 g_2)\,\zeta = g_1(g_2\zeta).$$

Definition

The group $U(E_c)$ is called the infinite dimensional unitary group. Sometime $U(E_c)$ is called unitary group.

For any g in $U(E_c)$, the adjoint g* can be defined through the canonical form in such a way that

$$<z, g\zeta> = <g^* z, \zeta> \text{ for every } z \in E_C^*,\ \zeta \in E_c. \quad (3)$$

The operator g* is a linear isomorphism of E_c^*. The collection $\{g^*;\, g \in U(E_c)\}$ again forms a group, call it $U^*(E_C^*)$, with the relation

$$g_1^* g_2^* = (g_2 g_1)^*,$$

which shows that $U^*(E_c^*)$ is anti-isomorphic to the group $U(E_C)$. A relation between the complex white noise and the unitary group $U(E_c)$ which is illustrated as follows.

Theorem 1

For every $g \in U(E_c)$, it holds that

$$g^*.\ \nu = \nu. \quad (4)$$

Proof

Since the σ field $\mathcal{B}$ is generated by the cylinder subsets of E_c^*, it suffices to show that

$$\{<z, \zeta>; \zeta \in E_c\} \text{ and } \{<z, g\zeta>; \zeta \in E_c\}$$

have the same probability distribution. In other words, it is enough to prove that for any finite number of ζ_k's, 1 ≤ k≤n. (<z, ζ_1>,…, z, ζ_n>) and (<z, gζ_1>,…., <z, g ζ_n>) have the same probability distribution on R^{2n}. This assertion is, hower, an easy consequence of the definition of ν and of the expression (2).

Corollary

The operator U_g on $L^2(E_c^*, \nu)$ given by

$$U_g\varphi(z) = \varphi(g^* z),\ \varphi \in L^2(E_c^*, \nu) \quad (5)$$

is unitary, The collection $\mathcal{U}(E_c) = \{U_g;\, g \in (E_c)\}$ is a group isomorphic to $U^*(E_c^*)$.

A topology can be introduced to the unitary group $U(E_c)$ by the use of $\mathcal{U}(E_c)$.

III. SUBGROUPS OF $U(S_C)$

In this section the Fourier transform $\mathcal{F}$ is especially stated. In order to apply $\mathcal{F}$ to a sample function of the complex white noise, let $\mathcal{F}$ be in $U(E_C)$. It may or may not be satisfied according as the choice of the basic space E. Another requirement arises when infinitesimal generators of one-parameter subgroup of $U(E_C)$ are discussed, where differential operators are given on E_c. Thus the countable Hilbertian norms $\| \|_n$, n ≥ 1, determining the topology of E and E_c must be defined in such a way that

$$\|D^n \xi\| = \|\xi\|_n,\ \xi \in E, \quad (6)$$

with some second order differential operator D. Finally, D and $\mathcal{F}$ are formally commutative as,

$$D\mathcal{F} = \mathcal{F}D.$$

Elementary observations lead to take the Schwartz space S as the basic space E and

$$D = \frac{d^2}{du^2} - (u^2 + 1).$$

The countable norms defined by (6) with the above differential operator determines the usual topology of S. Thus it is proved that the following proposition.

Proposition 1

The Fourier transform $\mathcal{F}$ is a member of $U(S_c)$

With the choice of $E = S$, interesting subgroups of $U(S_c)$ can be found. As soon as the unitary group $U(S_c)$ is defined, one may think that finite dimensional unitary group $U(n)$ is embedded by a suitable choice of a finite dimensional subspace of S_c.

Consider the so called Fourier-Mehler transform. Let $\mathcal{F}_\theta$ be defined by:

$$(F_\theta\ \zeta)(u) = K_\theta \circ \zeta(u) \equiv \int K_\theta(u, v)\,\zeta(v)\,dv, \qquad (7)$$

where

$$K_\theta(u, v) = \{\pi(1 - e^{2i\theta})\}^{-1/2}$$

$$\exp\left[-\frac{i(u^2 + v^2)}{2\tan\theta} + \frac{iuv}{\sin\theta}\right]$$

The operator $\mathcal{F}_\theta$ is well defined except the cases $\theta \equiv 0, \pi/2, \pi, 3\pi/2$ (mod, 2π), respectively.

Proposition 2

$\mathcal{F}_\theta$ extends to a continuous, periodic, one-parameter subgroup of $U(S_c)$:

$$\mathcal{F}_\theta \cdot \mathcal{F}_{\theta'} = \mathcal{F}_{\theta+\theta'} = \mathcal{F}_{\theta''},\quad \theta'' \equiv \theta + \theta' \,(\text{mod. } 2\pi) \qquad (8)$$

$\mathcal{F}_\theta \to I$ (identity) as $\theta \to 0$.

Proof

Take the system

$$\xi_n(u) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} H_n(u) \exp\left[-\frac{u^2}{2}\right],$$

(H_n : Hermite polynomial), $n \geq 0$, as a complete orthonormal system in $L_c^2(R^1)$. Elementary computations prove that $\mathcal{F}_\theta$ with the results

$$(\mathcal{F}_\theta \xi_n)(u) = e^{in\theta} \xi_n(u),\ \theta \neq 0, \pi/2, \pi, 3\pi/2,\ n = 0, 1, \ldots. \qquad (9)$$

It is noted that each ξ_n belongs to S_c and that the ξ_n's form a complete orthonomrnal system in $L_c^2(R^1)$. Thus by using (9), $\mathcal{F}_\theta$ extends to a one-parameter group satisfying the relation (8) for every n and then prove that $\mathcal{F}_\theta$ acts on the entire space S_c homeomorpically. It is obvious that $\|\mathcal{F}_\theta \zeta\| = \|\zeta\|$ hold for every θ and ζ. Thus the assertion is proved.

Sometimes, $\mathcal{F}_\theta$ is called the Fourier Mehler transform. With a particular choices of, $\theta = \pi/2$ and $\theta = 3\pi/2$, the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are embedded in the one-parameter group $\{\mathcal{F}_\theta\}$, namely $\mathcal{F}_{\pi/2} = \mathcal{F}$, $\mathcal{F}_{3\pi/2} = \mathcal{F}^{-1}$. One can apply F_θ^* to any sample function z of the complex white noise as follows:

$$< \mathcal{F}_\theta^* z, \zeta> = <z, \mathcal{F}_\theta \zeta>,$$

in particular

$$< \mathcal{F}_\theta^* z, \zeta_n> = e^{-in\theta} <z, \zeta_n>,$$

which tells us certain sample function property of the complex white noise (or the complex Brownian motion).

The next one-parameter subgroup to be discussed here is the shift which is the most important one in probability theory. We define

$$S_t : \zeta(u) \to (S_t\ \zeta)(u) = \zeta(u - t),\ -\infty < t < \infty. \qquad (10)$$

It is obvious that $\{S_t ; -\infty < t < \infty\}$ is a one-parameter subgroup of $U(S_c)$ such that

$$S_t \in U(S_c),$$
$$S_t \cdot S_s = S_{t+s}.$$

Recall that the importance of the flow $\{T_t\}$ of the complex Brownian motion on the measure space (E_c^*,υ) which comes from the shift $\{S_t\}$ defined by (10).

Here $\{S_t\}$ is a one-parameter subgroup of $U(S_c)$. The flow $\{T_t\}$ is obtained by taking the adjoint of S_t: $T_t = S_t^*$. By the use of the Fourier transform a third one- parameter subgroup of $U(S_c)$ arises from the shift . The operator π_t is defined on S_c by

$$\pi_t = \mathcal{F} S_t \mathcal{F}^{-1}, -\infty < t < \infty, \qquad (11)$$

where $\mathcal{F}$ stands for the ordinary Fourier transform on $L_c^2(R^1)$. Namely, π_t is defined as the conjugate for S_t with respect to $\mathcal{F}$. Then it turns out that π_t is expressed in the form

$$(\pi_t \zeta)(u) = e^{iut}\,\zeta(u)$$

and that it acts on S_c homeomorphically.

Observing the Weyl commutation relation between $\{S_t\}$ and $\{\pi_t\}$, a fourth one-parameter subgroup $\{I_t\}$ is naturally introduced:

$$(I_t\ \zeta)(u) = e^{it}\zeta(u),\ -\infty < t < \infty, \qquad (12)$$

and

$$\pi_s S_t = I_{st} S_t \pi_s \qquad (13)$$

So four one-parameter subgroups of $U(S_c)$ with special emphasis on Fourier transform and the shift. have been discussed.

The one-parameter group of dilations can be formed by the operators

$$g_t : \zeta(u) \to (g_t\ \zeta)(u) = e^{t/2}\,\zeta(e^t u),$$
$$-\infty < t < \infty. \qquad (14)$$

Here $\{g_t ; -\infty < t < \infty\}$ is also a one-parameter subgroup of $U(S_c)$. It is interesting that the flow of the complex Brownian motion arises as a transversal flow to $\{g_t^*\}$ on the measure space (S_c^*, ν) In terms of S_t and g_s $S_t g_s = g_s S_{t \exp(s)}$ is obtained.

So five one-parameter subgroups of $U(S_c)$ have been observed. For further heuristic approach it is convenient for us to use the infinitesimal generators of one-parameter subgroups. Such an approach will be discussed in the next section.

IV. Infinitesimal Generators and their Commutation Relations

The infinitesimal generator of a one-parameter group $\{g_t\}$ is defined by $\frac{d}{dt} g_t |_{t=0}$ with regardless its domain. The infinitesimal generator of every one-parameter subgroup of $U(S_c)$ which is considered that appeared in this section. Simple computations give the following table:

One-parameter group generator

$\mathcal{F}_\theta$ (Fourier-Mehler transform $if = -\frac{i}{2}\left(\frac{d^2}{du^2} - u^2 + 1\right)$

$$S_t \text{ (shift)} \quad s = -\frac{d}{du} \tag{15}$$

π_t (multiplication) $i\pi = iu$

g_t (dilation) $\tau = u\frac{d}{du} + \frac{1}{2}$

I_t (gauge transform) iI

The domain of each generator shown above is the entire space S_c.

Let [A , B] denote the commutator of A and B : [A , B] = AB − BA. To describe the commutation relations for the above generators, a new operator σ is introduced which is defined by $\sigma = \frac{1}{2}[\tau, f] = \frac{1}{2}\left(\frac{d^2}{du^2} + u^2\right)$. A one-parameter subgroup of $U(S_c)$ which can not be found because whose infinitesimal generator is σ, while $\exp[it\sigma]$ has certain physical meaning. The proof of the following relations is quite elementary.

$$\begin{aligned} &[s, f] = -\pi && [\tau, f] = 2\sigma \\ &[\pi, f] = -s && [\sigma, f] = 2\tau \\ &[\pi, s] = I && [\tau, \sigma] = 2f + I \\ &[\tau, s] = -s && [\sigma, s] = \pi \\ &[\tau, \pi] = \pi && [\sigma, \pi] = -s \end{aligned} \tag{16}$$

The algebraic property for these generators is quite simple as is indicated in the following theorem. The commutation relations (16) tell us many interesting properties of the correspond flows on (S_c^*, ν).

For example, some flow is a transversal flow to some other one; f replaces shift and multiplication with each other, and so forth. The following theorem can be proved.

Theorem 2

The vector space $\mathcal{A}$ spanned by $\{I, s, \pi, f, \tau, \sigma\}$ is closed under the product [,], that is, $\mathcal{A}$ forms a Lei algeba.

V. Conclusion

A subgroup of $U(S_c)$ consisting of all finite dimensional unitary transformations is also interesting to be investigated. For one thing, such a group has close connections with the infinite dimensional Laplacian operator or with the infinite dimensional harmonic function.

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