Matrix decompositions for quaternions

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***Abstract*—Since quaternions have isomorphic representations in matrix form we investigate various well known matrix decompositions for quaternions.**

***Keywords*—Decompositions of quaternions, Schur, polar, SVD, Jordan, QR, LU.**

I. INTRODUCTION

We will study various decompositions of quaternions where we will employ the isomorphic matrix images of quaternions. The matrix decompositions allow in many cases analogue decompositions of the underlying quaternion.

Let us denote the skew field of quaternions by $\mathbb{H}$. Let $a := (a_1, a_2, a_3, a_4) \in \mathbb{H}$. It is well known that a quaternion has an isomorphic representation either by a complex (2×2)-matrix, called the complex q-matrix, of the form

$$\tau(a) := \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathbb{C}^{2\times 2}, \quad \alpha := a_1 + a_2\mathbf{i},\ \beta := a_3 + a_4\mathbf{i},$$

or by a real (4×4)-matrix, called a real q-matrix,

$$\omega(a) := \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & -a_4 & a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{pmatrix} \in \mathbb{R}^{4\times 4},$$

where $\tau : \mathbb{H} \to \mathbb{C}^{2\times 2}$ and $\omega : \mathbb{H} \to \mathbb{R}^{4\times 4}$ are the corresponding isomorphisms.

The set of all complex q-matrices will be denoted by $\mathbb{H}_\mathbb{C}$. The set of all real q-matrices will be denoted by $\mathbb{H}_\mathbb{R}$..

II. RELATED DECOMPOSITIONS

We will study the possibility of decomposing quaternions with respect to various well known matrix decompositions:

- Schur decomposition: Let a be a quaternion, then we might ask whether there is a Schur decomposition of the matrices $\tau(a)$, $\omega(a)$ in terms of quaternions. If we formulate this problem for complex q-matrices we have to ask whether a decomposition of the following form is possible:

$$\begin{pmatrix} \sigma_+ & 0 \\ 0 & \sigma_- \end{pmatrix} =: \begin{pmatrix} \sigma & 0 \\ 0 & \overline{\sigma} \end{pmatrix} = \mathbf{U}^*\mathbf{A}\mathbf{U} =$$

$$\begin{pmatrix} \overline{u} & -v \\ \overline{v} & u \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix},$$

where σ_+ and σ_- are eigenvalues of the matrix $\tau(a)$, α, β are arbitrary, given complex numbers and u, v are wanted complex numbers such that $|u|^2 + |v|^2 = 1$. If we rewrite this equation with quaternions, it reads

$$\sigma = \overline{\mathbf{u}}a\mathbf{u}, \quad |\mathbf{u}| = 1,$$

where $\mathbf{u}$ is the quaternion defining the corresponding complex q-matrix $\mathbf{U}$. Let $a := (a_1, a_2, a_3, a_4) \in \mathbb{H}\backslash\mathbb{R}$, $\sigma := \sigma_+$ be the complex representative of a, see [5]. Then the last equation is the (complex) Schur decomposition of a.

The real Schur decomposition of a is analogical.

- The polar decomposition: The aim is to generalize the polar representation of a complex number. For matrices $\mathbf{A} \in \mathbb{C}^{m\times n}$, $m \leq n$, we are looking for a representation of the form $\mathbf{A} = \mathbf{P}\mathbf{U}$, where $\mathbf{P}$ is positive semidefinite and $\mathbf{U}$ is unitary.

Let $a := (a_1, a_2, a_3, a_4) \in \mathbb{H}\backslash\{0\}$. Its representation in terms of quaternions is

$$a = |a|\frac{a}{|a|}.$$

The corresponding matrices $\tau(a)$ and $\omega(a)$ are nonsingular square matrices with columns orthogonal to each other,

$$\tau(a) = \operatorname{diag}(|a|, |a|)\, \tau(\frac{a}{|a|})\,,$$

$$\omega(a) = \operatorname{diag}(|a|, |a|, |a|, |a|)\, \omega(\frac{a}{|a|})\,.$$

The first factor is positive definite and the second one is unitary or orthogonal.

From a purely algebraic standpoint this representation of a is complete. However, already the name polar representation means more. In the quaternionic case one finds, see for example [1],

$$\frac{a}{|a|} = \exp(\alpha u), \quad a \neq 0,$$

with $u := a_v/|a_v|, \alpha := \arctan(|a_v|/a_1)$, $a_v := (0, a_2, a_3, a_4)$ is so called vector part of a, and exp is defined by its Taylor series using $u^2 = -1$.

- The singular value decomposition: The wanted decomposition must be of the form

$$\begin{pmatrix} |a| & 0 \\ 0 & |a| \end{pmatrix} = \mathbf{U} \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \mathbf{V}^*$$

and the main question is whether $\mathbf{U}, \mathbf{V} \in \mathbb{H}_\mathbb{C}$. In order to solve this problem, we write the last equation directly in terms of quaternions,

$$|a| = ua\overline{v}, \qquad |u| = |v| = 1.$$

Let $a \in \mathbb{H}\backslash\mathbb{R}$. Choose $u \in \mathbb{H}$ with $|u| = 1$ and define $v := ua/|a|$ or, equivalently, choose v with $|v| = 1$ and define

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$u := v\overline{a}/|a|$. Then the last equation defines a singular value decomposition of a and

$$\tau(|a|) = \tau(u)\tau(a)\tau(v)^*$$

defines the corresponding SVD in $\mathbb{H}_\mathbb{C}$. A SVD with $u = v$ is impossible. The corresponding SVD in $\mathbb{H}_\mathbb{R}$ is

$$\omega(|a|) = \omega(u)\omega(a)\omega(v)^\mathrm{T}.$$

- The Jordan decomposition: Let $a := (a_1, a_2, a_3, a_4) \in \mathbb{H}\backslash\mathbb{R}$. Since the two eigenvalues $\sigma_\pm$ of $\tau(a)$ are different there will be an $s \in \mathbb{H}\backslash\{0\}$ such that $a = s^{-1}\sigma_+ s$ which implies

$$\tau(a) = \tau(s^{-1})\tau(\sigma_+)\tau(s).$$

And this representation is the Jordan decomposition of $\tau(a)$ and $\mathbf{J} := \tau(\sigma_+) = \begin{pmatrix} \sigma_+ & 0 \\ 0 & \sigma_- \end{pmatrix}$ is the Jordan canonical form of $\tau(a)$.

- The QR decomposition: All triangular matrices in $\mathbb{H}_\mathbb{C}$, and in $\mathbb{H}_\mathbb{R}$ reduce to diagonal matrices. Therefore, the QR-decomposition of a quaternion $a \neq 0$ has the trivial form

$$a = \frac{a}{|a|}\,|a| \iff \tau(a) = \tau\big(\frac{a}{|a|}\big)\,\tau(|a|), \quad \omega(a) = \omega\big(\frac{a}{|a|}\big)\,\omega(|a|),$$

which is identical with the polar decomposition.

- The LU decomposition: Since triangular matrices in $\mathbb{H}_\mathbb{C}$, and in $\mathbb{H}_\mathbb{R}$ reduce to diagonal matrices and since a product of two diagonal matrices is again diagonal an LU-decomposition of a quaternion a will in general not exist since $\tau(a), \omega(a)$ are in general not diagonal. So we may ask for the ordinary LU-decomposion of $\tau(a)$ and $\omega(a)$.

Let $a = (a_1, a_2, a_3, a_4) \in \mathbb{H}$. Put $\alpha := a_1 + a_2\mathbf{i}$ and $\beta := a_3 + a_4\mathbf{i}$. An LU decomposition of $\tau(a)$ exists if and only if $\alpha \neq 0$. If this condition is valid, then

$$\tau(a) \;=\; \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & u_{22} \end{pmatrix},$$

where

$$l_{21} \;=\; -\frac{\overline{\beta}}{\alpha}, \qquad u_{22} = \frac{|\alpha|^2 + |\beta|^2}{\alpha} = \frac{|a|^2}{\alpha}.$$

A Cholesky decomposition cannot be achieved since matrices $\tau(a)$ and $\omega(a)$ are missing symmetry.

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