Numerical Solution of One Nonlocal Mixed Problem

G.Mehdiyeva, A.Aliyev

Abstract—In this paper a mixed problem with nonlocal boundary conditions on a rectangular domain is considered. Error is estimated effectively, i.e. only the known problem data participate in this estimate.By applying the net method error estimates of approximate solution of the Laplace equation usually contain maximums of modulus of derivatives of the desired solution. And this naturally makes difficult to use estimates in practice. Error estimates of some methods expressed by basic problem data are known in the references. So Wazov [2] has estimated error of the Fourier discrete method for the Dirichlet problem of the Laplace equation by the paper [1]. As well E.A. Volkov as distinct from these papers applying the Gershgorin majorant method, summability method on layers has estimated error also only with the known data.

Keywords—Nonlocal, mixed, numerical, difference scheme, rectangular

I. INTRODUCTION

IN this paper first for nonlocal mixed problem, error of the Fourier discrete method is estimated effectively, i.e. error is estimated with the help of the known data.

The following nonlocal mixed problem is considered. It is required to find a solution of the equation

$$\Delta u = 0 \text{ on } \Pi, \tag{1}$$

satisfying the boundary conditions

$$\frac{\partial u}{\partial x} = 0 \text{ on } \Gamma_2,\ \Gamma_4, \tag{2}$$

$$u = 0 \text{ on } \Gamma_3$$

$$u(x,c) = \alpha u(x,c) + f(x), \quad (0 < x < 1) \tag{4}$$

where it is assumed that $f(x)$ is thrice continuously differentiable and $f'(0) = f'(1) = 0$.

Here $\Pi = \{(x,y): 0 < x < 1, 0 < y < b\}$, and Γ_i $(i = \overline{1,4})$ are the sides of the rectangular Π numbered counterclockwise starting with lower-side except for the ends. Introduce the following notation:

$$\Gamma = \bigcup_{i=1}^{4} \Gamma_i, \quad \overline{\Pi} = \Pi \cup \Gamma$$

Introduce quadratic net by straight lines $x = x_i = ih,\ y = y_j = jh\ (i,j = \overline{1,n})$. Denote

$$\Pi_h = \{(x,y): x = x_i = ih,\ y = y_j = jh,\ i,j = \overline{0,n}\},\ \Gamma_{ih}\ (i = \overline{1,4})$$

is a set of net nodes lying on Γ_i and $\Gamma_h = \bigcup_{i=1}^{4} \Gamma_{ih}$ respectively, as well as $\overline{\Pi}_h = \Pi_h \cup \Gamma_h$.

Let's construct the difference scheme of corresponding problem (1) – (4) in the following form:

$$\Delta_h u_h = 0 \text{ on } \Pi_h \tag{5}$$

$$-\frac{2}{h} u_{\overline{x}} + u_{\overline{y}y} = 0 \quad on \quad \Gamma_{2h}, \quad \frac{2}{h} u_{\overline{x}} + u_{\overline{y}y} = 0 \quad on \quad \Gamma_{4h} \tag{6}$$

$$u_h = 0 \quad on \quad \Gamma_{3h}, \tag{7}$$

$$u_h(x,0) = \alpha u_h(x,c) + f_h, \quad (\alpha < 1) \tag{8}$$

Here it is assumed that the point c coincides with on of nodes.

We prove that the solutions of problem (1) – (4) and (5) – (8) are defined by the following formulae respectively

$$u(x,y) = -\frac{a_0}{2[(1-\alpha)b + ac]}(y-b) + \sum_{n=1}^{\infty} a_n g(y, n\pi) \cos n\pi x, \tag{9}$$

$$u_h(x,y) = -\frac{b_0}{2[(1-\alpha)b + ac]}(y-b) + \sum_{n=1}^{1/h} b_n g(y, \frac{\beta_n}{h}) \cos n\pi x, \tag{10}$$

$$a_n = 2\int_0^1 f(t) \cos n\pi t dt, \quad a_0 = \int_0^1 f(t) dt,$$

$$b_n = 2h \sum_{k=1}^{1/h} f_h(kh) \cos n\pi kh, \quad b_0 = h \sum_{k=1}^{1/h} f_h(kh),$$

$$g(y,z) = \frac{shz(b-y)}{shzb - \alpha shz(b-c)}, \tag{11}$$

$$sh\frac{\beta_n}{2} = \sin\frac{nh\pi}{2}.$$

At first we prove formula (9). We'll search a solution in the form

$$u(x,y) = X(x) \cdot Y(y).$$

Then

$$X'' - kX = 0, \tag{12}$$

$$Y'' + kY = 0, \tag{13}$$

where k is some constant.

In order the function $u(x,y)$ satisfy boundary conditions (2) the function $X(x)$ must satisfy the conditions

$$X'(0) = X'(1) = 0 \tag{14}$$

We must find a solution of equation (12) satisfying condition (14). If assume $k = -\lambda^2$, then the general solution of equation (12) will be

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x.$$

Condition (14) gives

$$X'(0) = \lambda C_2 = 0,$$

$$X'(1) = -\lambda C_1 \sin \lambda + \lambda C_2 \cos \lambda = 0.$$

G.Yu.Mehdiyeva - doctor of science, PhD, professor, head of chair of Computational mathematics of Baku State University, Baku, Azerbaijan (corresponding author to provide phone 994125106048 e-mail: imn_bsu@mail.ru)

A.Aliyev - PhD, assos.prof. of the department of Computational mathematics of Baku State University, Baku, Azerbaijan (e-mail: aydin_aliyev66@yahoo.com) .

This work was supported by the Science Development Foundation of Azerbaijan (Grand EIF-2011-1(3)).

Consequently, $C_2 = 0$ certainly and we can take C_1 being equal to zero only provided

$$\sin\lambda = 0,$$

i.e. if λ is a number divisible by π :

$$\lambda = n\pi \qquad (n = 1,2,...).$$

At $\lambda = n\pi$ we obtain the solution

$$X(x) = C\cos n\pi x \qquad (n = 1,2,...).$$

Substituting $k = -\lambda^2 = -(n\pi)^2$ in (13) we obtain for Y the equation

$$Y'' - (n\pi)^2 Y = 0$$

whose solution is defined by the formula

$$Y = C_1 chn\pi y + C_2 shn\pi y\,,$$

where C_1, C_2 are arbitrary constants

$$u(x,y) = \sum_{n=1}^{\infty}[A_n chn\pi y + B_n shn\pi y]\cos n\pi x + A_0 y + B_0\,.$$

This function satisfies equation (1) and boundary conditions (2), it remains to select the constants A_n and B_n such that to satisfy boundary condition (3) and (4).

According to these solutions

$$\sum_{n=1}^{\infty}[A_n chn\pi b + B_n shn\pi b]\cos n\pi x + A_0 b + B_0 = 0,$$

$$\sum_{n=1}^{\infty} A_n \cos n\pi x + B_0 =$$

$$= \alpha\{\sum_{n=1}^{\infty}[A_n chn\pi c + B_n shn\pi c]\cos n\pi x + A_0 c + B_0\} + f$$

On the other hand subject to

$$f(x) = \sum_{n=1}^{\infty} a_n \cos n\pi x + \frac{a_0}{2},$$

where

$$a_n = 2\int_0^1 f(x)\cos n\pi x dx, \quad a_0 = \int_0^1 f(x)dx.$$

Hence comparing the coefficients of series we obtain

$$A_0 = -\frac{a_0}{2[(1-\alpha)b + \alpha c]}, \quad B_0 = -\frac{b_0}{2[(1-\alpha)b + \alpha c]}$$

$$A_n = -\frac{shn\pi b}{(1-\alpha chn\pi c)shn\pi b + \alpha shn\pi c \cdot chn\pi b} \cdot a_n\,,$$

$$B_n = -\frac{chn\pi b}{(1-\alpha chn\pi c)shn\pi b + \alpha shn\pi c \cdot chn\pi b} \cdot a_n$$

Thus

$$u(x,y) = -\frac{a_0}{2[(1-\alpha)b + \alpha c]}(y-b) +$$

$$+\sum_{n=1}^{\infty} a_n \frac{shn\pi(b-y)}{shn\pi b - \alpha shn\pi(b-c)} \cdot \cos n\pi x.$$

By immediate testing it is easy to be convinced that $u_h(x,y)$ defined by formula (10) is a solution of problem (5) – (8).

It is known that ([1])

$$a_n = b_n \quad (n = 0,1,...,\frac{1}{h})\,.$$

We'll get the solution of difference scheme (5) – (8) $u_h(x,y)$ as a approximate solution of problem (1) – (4).

We estimate error of method. From (9) and (10) we have

$$|u - u_h| \le R_1 + R_2,$$

where

$$R_1 = \sum_{n=1}^{1/h} |b_n|\,|g(y,n\pi) - g(y,\frac{\beta_n}{h})|, \qquad R_2 = \sum_{n=1+1/h}^{1/h} |a_n|\,|g(y,n\pi)|\,.$$

We estimate $|a_n|$. Using the formula of integration by parts (twice) we obtain

$$|a_n| \le \frac{4}{(n\pi)^4}\max|f'''(t)| = kn^{-4}\,, \qquad (15)$$

where

$$k = \frac{4}{\pi^4}\max|f'''(x)|\,.$$

Hence it follows that

$$|b_n| \le kn^{-4} \qquad (n = 1,2,...,1/h). \qquad (16)$$

Consequently, in order to estimate R_i $(i = 1,2)$ it is necessary to estimate

$$|g(y,z)| \text{ and } \left|g(y,n\pi) - g\left(y,\frac{\beta_n}{h}\right)\right|.$$

It is easy to note that

$$|g(y,z)| \le \frac{1}{1-\alpha}\,. \qquad (17)$$

We estimate $\left|g(y,n\pi) - g\left(y,\frac{\beta_n}{h}\right)\right|$. We have

$$\frac{\partial g}{\partial z} = \frac{1}{2}[shbz - \alpha sh(b-c)z]^{-2} \cdot \{(2b-y)shyz - ysh(2b-y)z -$$

$$-\alpha[(2(b-c)-(y-c))sh(y-c)z - (y-c)sh(2(b-c)-(y-c))z]\}.$$

Subject to

$$(2b-y)shyz \le y \cdot sh(2b-y)z$$

and

$$(2(b-c)-(y-c))sh(y-c)z \le (y-c)sh(2(b-c)x-(y-c))z.$$

We obtain

$$\left|\frac{\partial g(y,z)}{\partial z}\right| \le \frac{1}{2}[shbz - \alpha sh(b-c)z]^{-2} \cdot [ysh(2b-y)z +$$

$$+\alpha|y-c|sh(2b-c-y)z]. \qquad (18)$$

We'll use below the following obvious inequalities

$$shkt \le \exp((k-1)t)sht \qquad (0 \le k \le 1)\,, \quad (19)$$

$$sht \ge \frac{1}{2}(1-\exp(-2t_1))\exp(t) \quad (t \ge t_1 > 0)\,, \quad (20)$$

$$\frac{4}{3}n \le \frac{\beta_n}{h} \le n\pi \quad (1 \le n \le \frac{1}{h})\,. \qquad (21)$$

Using formula (19) and assuming

$$0 \le k = 1 - \frac{y}{2b} \le 1, \;\; t = 2b \cdot z\,,$$

$$0 \le k = 1 - \frac{c+y}{2b} \le 1, \;\; t = 2bz$$

respectively we obtain

$$sh(2b-y)z = sh\left(1-\frac{y}{2b}\right)2bz \le \exp(-yz)sh2bz\,, \quad (22)$$

$$sh(2b-c-y)z = sh\left(1-\frac{c+y}{2b}\right)2bz \le \exp(-(c+y)z)sh2bz \tag{23}$$

Now assuming

$$t = bz,\ t_1 = \frac{4}{3}bn$$

in (20) at $z \ge \frac{\beta_n}{h}$ using formula (21) we obtain

$$shbz \ge \frac{1}{2}(1-\exp(-\frac{8bn}{3}))\exp(bz)\ .$$

Hence at $n \ge 1$ and $z \ge \frac{\beta_n}{h} \ge \frac{4}{3}n$ we obtain

$$[shbz - \alpha \cdot sh(b-c)z]^{-2} \le \left[\frac{1}{2}\left(1-\exp\left(-\frac{8b}{3}\right)\right)\exp(bz) -\right.$$

$$\left. -\alpha sh(b-c)z\right]^{-2} = \left[\frac{1}{2}\left(1-\exp\left(-\frac{8b}{3}\right)\right) -\right.$$

$$\left. -\alpha\exp(-bz)sh(b-c)z\right]^{-2} \cdot \exp(-2bz) \le$$

$$\le 4\left[1-\exp\left(-\frac{8b}{3}\right)-\alpha\exp(-cz)\right]^{-2}\exp(-2bz) \le$$

$$\le 4\left[1-\exp\left(-\frac{8b}{3}\right)-\alpha\exp\left(-\frac{4c}{3}\right)\right]^{-2}\exp(-2bz). \tag{24}$$

Allowing for (22), (23) and (24) in (18) obtain

$$\left|\frac{\partial g(y,z)}{\partial z}\right| \le 2\left(1-\exp\left(-\frac{8b}{3}\right)-\alpha\exp\left(-\frac{4c}{3}\right)\right)^{-2} \times$$

$$\times\exp(-2bz)[y\exp(-yz)sh2bz +$$

$$+\alpha|y-c|\exp(-(c+y)z)sh2bz] \le$$

$$\le 2\left(1-\exp\left(-\frac{8b}{3}\right)-\alpha\exp\left(-\frac{4c}{3}\right)\right)^{-2} \times$$

$$\times[y\exp(-yz)+\alpha(y-c)\exp(-(c+y)z)]\times$$

$$\times\exp(-2bz)sh2bz \le \left(1-\exp\left(-\frac{8b}{3}\right)-\alpha\exp\left(-\frac{4c}{3}\right)\right)^{-2} \times$$

$$\times[y\exp(-yz)+\alpha(y-c)\exp(-(c+y)z)].$$

Thus, at $1 \le n \le 1/h$ and $\beta_n/h < z < n\pi, 0 \le y \le b$ we obtain

$$\left|\frac{\partial g(y,z)}{\partial z}\right| \le \left(1-\exp\left(-\frac{8b}{3}\right)-\alpha\exp\left(-\frac{4}{3}\right)\right)^{-2} \times$$

$$\times[y\exp(-yz)+\alpha(y+c)\exp(-(c+y)z)].$$

Then

$$\left|g\left(y,\frac{\beta_n}{h}\right)-g(y,n\pi)\right| \le$$

$$\le\left(1-\exp\left(-\frac{8b}{3}\right)-\alpha\exp\left(-\frac{4}{3}\right)\right)^{-2}[y\cdot\exp(-yz)+$$

$$+\alpha(y+c)\exp(-(c+y)z)]\frac{(nh\pi)^3}{6h} \le$$

$$\le\frac{\pi^3}{6}n^3\left(1-\exp\left(-\frac{8b}{3}\right)-\alpha\exp\left(-\frac{4}{3}\right)\right)^{-2} \times$$

$$\times\left[y\exp\left(-\frac{4}{3}yn\right)+\alpha(y+c)\exp\left(-\frac{4}{3}(y+c)n\right)\right]h^2 \tag{25}$$

It follows from (16) and (17)

$$R_2 \le k\sum_{n=1+\frac{1}{h}}^{\infty} n^{-4} \le k\frac{h^3}{3}.$$

It follows from (15) and (25) that

$$|R_1| \le k\frac{\pi^3}{6}\left\{\sum_{n=1}^{1/h}(n^{-4}n^3)\left[y\exp\left(-\frac{4y}{3}n\right)+\alpha(y+c)\exp\left(-\frac{4(y+c)}{3}n\right)\right]\right\}\times$$

$$\times\left(1-\exp\left(-\frac{8b}{3}\right)-\alpha\exp\left(-\frac{4c}{3}\right)\right)^{-2}h^2 =$$

$$= k\frac{\pi^3}{6}\left(1-\exp\left(-\frac{8b}{3}\right)-\alpha\exp\left(-\frac{4c}{3}\right)\right)^{-2}h^2 =$$

$$= k\frac{\pi^3}{6}\left(1-\exp\left(-\frac{8b}{3}\right)-\alpha\exp\left(-\frac{4c}{3}\right)\right)^{-2}\times$$

$$\times\left[y\sum_{n=1}^{1/h}\left(\exp\left(-\frac{4y}{3}\right)\right)^n+\alpha(y+c)\sum_{n=1}^{1/h}\left(\exp\left(-\frac{4(y+c)}{3}\right)\right)^n\right] \le$$

$$\le\frac{1}{6}k\pi^3h^2\left(1-\exp\left(-\frac{8b}{3}\right)-\alpha\exp\left(-\frac{4c}{3}\right)\right)^{-2}\times$$

$$\times\left[\frac{y\exp\left(-\frac{4y}{3}\right)}{1-\exp\left(\frac{4y}{3}\right)}+\alpha\frac{(y+c)\exp\left(-\frac{4(y+c)}{3}\right)}{1-\exp\left(\frac{4(y+c)}{3}\right)}\right] \le$$

$$\le k\frac{\pi^3}{8}(1+\alpha)\left(1-\exp\left(-\frac{8b}{3}\right)-\alpha\exp\left(-\frac{4c}{3}\right)\right)^{-2}h^2.$$

Thus

$$|u-u_h| \le k\left\{\frac{1}{3}+\frac{1}{8}\pi^3\left(1-\exp\left(-\frac{8b}{3}\right)-\alpha\exp\left(-\frac{4c}{3}\right)\right)^{-2}(1+\alpha)\right\}h^2$$

REFERENCES

[1] Walsh J.L., Young D. On the accuracy of the numerical solution of the Dirichlet problem by finite differences. Jour. of Resea. of the Nat. Bur. of Standards, 1953, V.51,No.6, pp.343-369.

[2] Wasow W. On the truncation error in the solution of Laplace's equation by finite differences. Jour. of Resea. of the Nat. Bur. of Standards, 1952, V.48, pp.345-348.

[3] Giese J.H. On the truncation error in a numerical solution of the Neumann problem for a regtangle. Jour. Math. and Phys., 1958, v.37, No.2, pp. 169-177

[4] Romanova S.E. Economical method of solution of Laplace's difference equation on rectangle domains. DAN SSSR, 1980, v.252, No. 1, pp. 48-51 (Russian)

[5] Romanova S.E. Economical methods of approximated solution of Laplace's difference equation on rectangle domains. Zh. vuch.mat. i mat. fiz., 1983, v.23, No.3, pp.660-673 (Russian)

[6] Aliyev A.Yu. On numerical solution nonlocal boundary values problems for elliptic equations. Ph. D. thesis, Baku, 1992 (Russian)

[7] Aliyev A.Yu. Efficient error estimate of net method for a mixed boundary value problem. Transactions Azer. Nat. Acad of Sciences., 2003, v.23, No. 4, pp.233-238

[8] Aliyev A.Yu. Numerical solution of one nonhomogeneous nonlocal mixed problem. Proceedings of Institute of Mat. and Mech. Nat. Acad. of Sciences of Azerbaijan, 2005, v. 22, pp. 179-186.