Spectral radius of some special nonnegative matrices

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Abstract—In this paper for two X-form nonnegative matrices A and B we show that if $A - B$ be a symmetric positive semi-definite matrix then the spectral radius of matrix A greater than or equal matrix B.

Keywords—Nonnegative matrix, Spectral radius.

I. Introduction

A matrix A is called nonnegative X-form matrix, if all its nonzero elements are nonnegative and placed on main diagonal or anti-diagonal of matrix A. In recent paper [1] Jonathan Axtell, et al. showed that when A is symmetric matrix and S_1 and S_2 are two symmetric matrices such that the difference $S_1 - S_2$ is a positive semidefinite matrix and such that S_1A and S_2A are nonnegative matrices, then

$$\rho(S_2A) \leq \rho(S_1A).$$

In this paper we try without considering two symmetric positive definite matrices S_1 and $_2$ obtain the inequality relation for spectral radius of two special matrices A and B. At first for two 2×2 nonnegative matrix A and B where $A - B$ is symmetric positive semidefinite, we show that $\rho(A) \geq \rho(B)$, and then we show for two nonnegative X-form matrix this property also holds. By considering [1], at last we verify for two normal matrix A and B with $A - B \geq 0$ that satisfy in condition $AB = BA$, we have

$$\rho(A) \geq \rho(B).$$

II. Main results

Lemma 2.1: If A and B be 2×2 nonnegative matrices and $A - B \geq 0$, then $\rho(A) \geq \rho(B)$.

Proof. Let $S = A - B$ and A, B are nonnegative matrices as blow

$$S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad B = \begin{pmatrix} m & n \\ p & q \end{pmatrix},$$

$$A = \begin{pmatrix} a+m & b+n \\ b+p & c+q \end{pmatrix},$$

whereas A is nonnegative matrix, then

$$m \geq -a, \qquad n \geq -b, \qquad p \geq -b, \qquad q \geq -c, \tag{1}$$

since S is symmetric positive semi-definite, we have $X^TSX \geq 0$ for all vectors X. By choosing special vector $X = [1, 0]$ and $X = [0, 1]$ we obtain

$$a \geq 0,\ c \geq 0 \Rightarrow a + c \geq 0, \tag{2}$$

and again since S is symmetric positive semi-definite matrix, then

$$\det(S) = ac - b^2 \geq 0. \tag{3}$$

Now we compare $\rho(A)$ and $\rho(B)$ and show that $\rho(A) \geq \rho(B)$.

$$\det(A - \lambda I) = 0 \Rightarrow \lambda = \tfrac{1}{2}((a+m+c+q) \pm \sqrt{(a+m+c+q)^2 - 4(a+m)(c+q) + 4(b+p)(b+n)}).$$

Consequently

$$\rho(A) = \tfrac{1}{2}((a+m+c+q) + \sqrt{(a+m+c+q)^2 - 4(a+m)(c+q) + 4(b+p)(b+n)}) \tag{4}$$

and from $\det(B - \lambda I) = 0$, similarly we have

$$\rho(B) = \frac{1}{2}\left((m+q) + \sqrt{(m+q)^2 - 4mq + 4np}\right). \tag{5}$$

Whereas for all two real numbers the geometrical mean less than or equal the arithmetical mean, we have

$$\sqrt{(a+m)(c+q)} \leq \tfrac{a+m+c+q}{2} \Rightarrow (a+m+c+q)^2 - 4(a+m)(c+q) \geq 0,$$
$$\sqrt{mq} \leq \tfrac{m+q}{2} \Rightarrow (m+q)^2 - 4mq \geq 0,$$

then the relations (4) and (5) is well define.
Now, we show that $\rho(A) \geq \rho(B)$.
Since $(a + c) \geq 0$, then $(a + m + c + q) \geq (m + q)$ and it is sufficient that we show

$$\sqrt{(a+m+c+q)^2 - 4(a+m)(c+q) + 4(b+p)(b+n)} \geq \sqrt{(m+q)^2 - 4mq + 4np}.$$

This inequality holds if and only if

$$(a+m+c+q)^2 - 4(a+m)(c+q) + 4(b+p)(b+n) \geq (m+q)^2 - 4mq + 4np \Leftrightarrow$$

$$a^2 + c^2 - 2aq - 2mc + 2mq + 4b^2 + 4bn + 4bp \geq 0. \tag{6}$$

For proof (6) by (1) we can write

$$(a+m+c+q)^2 - 4(a+m)(c+q) + 4(b+p)(b+n) \geq a^2 + c^2 + 2ac + 2ac + 2ac + 4b^2 - 4b^2 + 4b^2 = (a+c)^2 + 4(ac - b^2) \geq 0,$$

So $\rho(A) \geq \rho(B)$.

Lemma 2.2: . If S be a $n \times n$ symmetric positive semi-definite matrix, then every 2×2 submatrix of S as below $S' = \begin{pmatrix} s_{ii} & s_{ij} \\ s_{ji} & s_{jj} \end{pmatrix}$ is also positive semi-definite.

Proof. Proof is trivial.

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Theorem 2.3: . If A and B be two $n \times n$, X-form nonnegative matrices such that $S = A - B \geq 0$, then $\rho(A) \geq \rho(B)$.

Proof. Let given two matrix A and B with even order in following form

$$A = \begin{pmatrix} a_1 & & & & & b_1 \\ & a_2 & & & b_2 & \\ & & \ddots & \iddots & & \\ & & \iddots & \ddots & & \\ & & & & & \\ b_n & & & & & a_n \end{pmatrix}_{n\times n},$$

$$B = \begin{pmatrix} a'_1 & & & & & b'_1 \\ & a'_2 & & & b'_2 & \\ & & \ddots & \iddots & & \\ & & \iddots & \ddots & & \\ & & & & & \\ b'_n & & & & & a'_n \end{pmatrix}_{n\times n},$$

then

$$S = A - B = \begin{pmatrix} c_1 & & & & & d_1 \\ & c_2 & & & d_2 & \\ & & \ddots & \iddots & & \\ & & \iddots & \ddots & & \\ & & & & & \\ d_n & & & & & c_n \end{pmatrix}_{n\times n},$$

where $c_i = a_i - b_i$ and $d_i = b_i - b'_i$ for $i = 1, 2, \ldots, n$. On the other hand we have

$$\det(A - \lambda I) = \prod_{i=1}^{n/2} \det\left(\begin{pmatrix} a_i & b_i \\ b_{n-i+1} & a_{n-i+1} \end{pmatrix} - \lambda I_2\right). \quad (7)$$

Assume

$$A_i = \begin{pmatrix} a_i & b_i \\ b_{n-i+1} & a_{n-i+1} \end{pmatrix}, \qquad B_i = \begin{pmatrix} a'_i & b'_i \\ b'_{n-i+1} & a'_{n-i+1} \end{pmatrix},$$

be two 2×2 matrices with elements of A and B respectively, then these matrices also are nonnegative.

If $S_i = A_i - B_i = \begin{pmatrix} c_i & d_i \\ d_{n-i+1} & c_{n-i+1} \end{pmatrix}$, then we have $S_i \geq 0$ and $\rho(A) \geq \rho(B)$ by Lemma 2.1.

By (4) the eigenvalue of A will be all eigenvalue of matrices A_i and the eigenvalue of B be all eigenvalue of matrices B_i for $i = 1, 2, \ldots, n/2$, therefore $\rho(A) \geq \rho(B)$.

If order of matrices A and B be an odd number, then the matrices A and B have the following form

$$A = \begin{pmatrix} a_1 & \cdots & \cdots & \cdots & b_1 \\ 0 & a_2 & \cdots & b_2 & 0 \\ \vdots & \vdots & a_{[n/2]+1} & \vdots & \vdots \\ b_{n-1} & 0 & \ldots & \ddots & a_{n-1} \end{pmatrix}_{n\times n}$$

$$B = \begin{pmatrix} a'_1 & \cdots & \cdots & \cdots & b'_1 \\ 0 & a'_2 & \cdots & b'_2 & 0 \\ \vdots & \vdots & a'_{[n/2]+1} & \vdots & \vdots \\ b'_{n-1} & 0 & \ldots & \ddots & a'_{n-1} \end{pmatrix}_{n\times n}$$

then

$$\det(A - \lambda I) = (a_{[n/2]+1} - \lambda) \prod_{i=1}^{[n/2]} \det(\begin{pmatrix} a_i & b_i \\ b_{n-i+1} & a_{n-i+1} \end{pmatrix} - \lambda I_2) \quad (8)$$

$$\det(B - \lambda I) = \prod_{i=1}^{n/2} \det(\begin{pmatrix} a'_i & b'_i \\ b'_{n-i+1} & a'_{n-i+1} \end{pmatrix} - \lambda I_2). \quad (9)$$

Since $a_{[n/2]+1} \geq a'_{[n/2]+1}$, then we have $\rho(A) \geq \rho(B)$.

Theorem 2.4: .If matrices A and B be normal and $A - B \geq 0$ and $AB = BA$, then $\rho(A) \geq \rho(B)$.

Proof. Since A and B are normal matrix and $AB = BA$ then there exist an unitary matrix U such that

$$A = U\mathrm{diag}(\lambda_1(A), \cdots, \lambda_n(A)U^*,$$
$$A = U\mathrm{diag}(\lambda_1(B), \cdots, \lambda_n(B)^n)U^*.$$

On the other hand $A - B$ also is normal matrix and

$$A - B = U\mathrm{diag}(\lambda_1(A), \cdots, \lambda_n(A))U^*$$

and since $A - B \geq 0$, then $\lambda_i(A) - \lambda_i(B) \geq 0$ for $i = 1, 2, \cdots, n$, then $\rho(A) \geq \rho(B)$.

References

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