

k -Fuzzy Ideals of Ternary Semirings

Sathinee Malee and Ronnason Chinram

Abstract—The notion of k -fuzzy ideals of semirings was introduced by Kim and Park in 1996. In 2003, Dutta and Kar introduced a notion of ternary semirings. This structure is a generalization of ternary rings and semirings. The main purpose of this paper is to introduce and study k -fuzzy ideals in ternary semirings analogous to k -fuzzy ideals in semirings considered by Kim and Park.

Keywords— k -ideals, k -fuzzy ideals, fuzzy k -ideals, ternary semirings

I. INTRODUCTION

The notion of ternary algebraic system was introduced by Lehmer [15] in 1932. He investigated certain ternary algebraic systems called triplexes. In 1971, Lister [16] characterized additive semigroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. Dutta and Kar [3] introduced a notion of ternary semirings which is a generalization of ternary rings and semirings, and they studied some properties of ternary semirings ([3], [4], [5], [6], [7] and [11], etc.).

The theory of fuzzy sets was first studied by Zadeh [17] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory, etc. Fuzzy ideals of semirings were studied by some authors ([1], [2], [8], [9], [10] and [14], etc.). The notion of k -fuzzy ideals of semirings was introduced by Kim and Park [14]. Recently, Kavikumar, Khamis and Jun studied fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals in ternary semirings in [12] and [13]. The fuzzy ideal of ternary semirings is a good tool for us to study the fuzzy algebraic structure. The main purpose of this paper is to study k -fuzzy ideals in ternary semirings analogous to k -fuzzy ideals in semirings considered by Kim and Park.

II. PRELIMINARIES

In this section, we refer to some elementary aspects of the theory of semirings and ternary semirings and fuzzy algebraic systems that are necessary for this paper.

Definition 2.1. A nonempty set S together with two associative binary operations called addition and multiplication (denoted

Sathinee Malee is with the Department of Mathematics, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla, 90110 THAILAND, e-mail:sathinee_e@hotmail.com

Ronnason Chinram is with the Department of Mathematics, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla, 90110 THAILAND and Centre of Excellence in Mathematics, CHE, Si Ayuthaya Road, Bangkok 10400, THAILAND, e-mail:ronnason.c@psu.ac.th.

Most of the work in this paper is a part of the Master thesis written by Miss Sathinee Malee under the supervision of Assistant Professor Dr. Ronnason Chinram.

This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

by $+$ and \cdot , respectively) is called a *semiring* if $(S, +)$ is a commutative semigroup, (S, \cdot) is a semigroup and multiplicative distributes over addition both from the left and from the right, i.e., $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in S$.

Definition 2.2. A nonempty set S together with a binary operation and a ternary operation called addition $+$ and ternary multiplication, respectively, is said to be a *ternary semiring* if $(S, +)$ is a commutative semigroup satisfying the following conditions: for all $a, b, c, d, e \in S$,

- (i) $(abc)de = a(bcd)e = ab(cde)$,
- (ii) $(a + b)cd = acd + bcd$,
- (iii) $a(b + c)d = abd + acd$ and
- (iv) $ab(c + d) = abc + abd$.

We can see that any semiring can be reduced to a ternary semiring. However, a ternary semiring does not necessarily reduce to a semiring by this example. We consider \mathbb{Z}_0^- , the set of all non-positive integers under usual addition and multiplication, we see that \mathbb{Z}_0^- is an additive semigroup which is closed under the triple multiplication but is not closed under the binary multiplication. Moreover, \mathbb{Z}_0^- is a ternary semiring but is not a semiring under usual addition and multiplication.

Definition 2.3. Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0 + x = x = x + 0$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$, then 0 is called the *zero element* or simply the *zero* of the ternary semiring S . In this case we say that S is a ternary semiring with zero.

Definition 2.4. An additive subsemigroup T of S is called a *ternary subsemiring* of S if $t_1 t_2 t_3 \in T$ for all $t_1, t_2, t_3 \in T$.

Definition 2.5. An additive subsemigroup I of S is called a *left [resp. right, lateral] ideal* of S if $s_1 s_2 i \in I$ [resp. $i s_1 s_2 \in I, s_1 i s_2 \in I$] for all $s_1, s_2 \in S$ and $i \in I$. If I is a left, right and lateral ideal of S , then I is called an *ideal* of S .

It is obvious that every ideal of a ternary semiring with zero contains a zero element.

Definition 2.6. Let S and R be ternary semirings. A mapping $\varphi : S \rightarrow R$ is said to be a *homomorphism* if $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(xyz) = \varphi(x)\varphi(y)\varphi(z)$ for all $x, y, z \in S$.

Let $\varphi : S \rightarrow R$ be an onto homomorphism of ternary semirings. Note that if I is an ideal of S , then $\varphi(I)$ is an ideal of R . If S and R be ternary semirings with zero 0 , then $\varphi(0) = 0$.

Definition 2.7. Let S be a non-empty set. A mapping $f : S \rightarrow [0, 1]$ is called a *fuzzy subset* of S .

Definition 2.8. Let A be a subset of a non-empty set S . The *characteristic function* χ_A of A is a fuzzy subset of S defined

as follows:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Definition 2.9. Let f be a fuzzy subset of a non-empty subset S . For $t \in [0, 1]$, the set $f_t = \{x \in S \mid f(x) \geq t\}$ is called a *level subset* of S with respect to f .

III. MAIN RESULT

Definition 3.1. An ideal I of a ternary semiring S is said to be a *k-ideal* if for $x, y \in S, x + y, y \in I \Rightarrow x \in I$.

Example 3.1. Consider the ternary semiring \mathbb{Z}_0^- under usual addition and ternary multiplication, let $I = \{0, -3\} \cup \{-5, -6, -7, \dots\}$. It is easy to prove that I is an ideal of \mathbb{Z}_0^- but not a k-ideal of \mathbb{Z}_0^- because $-3, (-2) + (-3) \in I$ but $-2 \notin I$.

Example 3.2. Consider the ternary semiring \mathbb{Z}_0^- under usual addition and ternary multiplication, let $I = \{-3k \mid k \in \mathbb{N} \cup \{0\}\}$. It is easy to show that I is a k-ideal of \mathbb{Z}_0^- .

Definition 3.2. For each ideal I of a ternary semiring S , the *k-closure* \bar{I} of I is defined by

$$\bar{I} = \{x \in S \mid a + x = b \text{ for some } a, b \in I\}.$$

The next theorem holds.

Theorem 3.1. Let I be an ideal of a ternary semiring S with zero. Then I is a k-ideal of S if and only if $I = \bar{I}$.

Definition 3.3. A fuzzy subset f of a ternary semiring S is called a *fuzzy ideal* of S if for all $x, y, z \in S$,

- (i) $f(x + y) \geq \min\{f(x), f(y)\}$ and
- (ii) $f(xyz) \geq \max\{f(x), f(y), f(z)\}$.

By the definitions of ideals and fuzzy ideals of ternary semirings, the following lemma holds.

Lemma 3.2. Let I be a non-empty subset of a ternary semiring S . Then I is an ideal of S if and only if the characteristic function χ_I is a fuzzy ideal of S .

Lemma 3.3. Let f be a fuzzy ideal of a ternary semiring S with zero 0. Then $f(x) \leq f(0)$ for all $x \in S$.

Proof. For any $x \in S, f(0) = f(00x) \geq \max\{f(0), f(x)\} \geq f(x)$. \square

Definition 3.4. A fuzzy ideal f of a ternary semiring S with zero 0 is said to be a *k-fuzzy ideal* of S if

$$f(x + y) = f(0) \text{ and } f(y) = f(0) \Rightarrow f(x) = f(0)$$

for all $x, y \in S$.

Example 3.3. Consider the ternary semiring \mathbb{Z}_0^- under usual addition and ternary multiplication. Define a fuzzy subset f on \mathbb{Z}_0^- by

$$f(x) = \begin{cases} 0 & \text{if } x = -1, \\ 0.5 & \text{otherwise.} \end{cases}$$

It is easy to prove that f is a fuzzy ideal of \mathbb{Z}_0^- . However, f is not a k-fuzzy ideal of \mathbb{Z}_0^- because $f((-1) + (-2)) =$

$f(-3) = 0.5 = f(0)$ and $f(-2) = 0.5 = f(0)$ but $f(-1) = 0 \neq 0.5 = f(0)$.

Example 3.4. Let f be a fuzzy subset of a ternary semiring \mathbb{Z}_0^- under usual addition and ternary multiplication defined by

$$f(x) = \begin{cases} 0.3 & \text{if } x \text{ is odd} \\ 0.5 & \text{if } x \text{ is even.} \end{cases}$$

It is easy to show that f is a fuzzy ideal of \mathbb{Z}_0^- . Let $x, y \in \mathbb{Z}_0^-$ such that $f(x+y) = f(0)$ and $f(x) = f(0)$. So $f(x+y) = 0.5$ and $f(y) = 0.5$. Thus $x + y$ and y are even. Hence x is even, this implies $f(x) = 0.5 = f(0)$. Therefore f is a k-fuzzy ideal of \mathbb{Z}_0^- .

From the condition of Definition 3.4 and Lemma 3.2, the following theorem holds.

Theorem 3.4. Let S be a ternary semiring with zero 0 and I is a non-empty subset of S . Then I is a k-ideal of S if and only if the characteristic function χ_I is a k-fuzzy ideal of S .

Proof. Assume I is a k-ideal of S . By Lemma 3.2, χ_I is a fuzzy ideal of S . Next, let $x, y \in S$ and assume $\chi_I(x + y) = \chi_I(0)$ and $\chi_I(y) = \chi_I(0)$. Since I is an ideal of $S, 0 \in I$. Thus $\chi_I(0) = 1$, this implies $\chi_I(x + y) = 1$ and $\chi_I(y) = 1$. Then $x + y, y \in I$. Since I is a k-ideal of $S, x \in I$. Hence $\chi_I(x) = 1 = \chi_I(0)$. Therefore χ_I is a k-fuzzy ideal of S . Conversely, assume characteristic function χ_I is a k-fuzzy ideal of S . By Lemma 3.2, I is an ideal of S . So $0 \in I$, this implies $\chi_I(0) = 1$. Let $x, y \in S$ such that $x + y, y \in I$. So $\chi_I(x + y) = \chi_I(0)$ and $\chi_I(y) = \chi_I(0)$. Then $\chi_I(x) = \chi_I(0) = 1$. So $x \in I$. Hence I is a k-ideal of S . \square

Theorem 3.5. Let f be a fuzzy subset of a ternary semiring S . Then f is a fuzzy ideal of S if and only if for any $t \in [0, 1]$ such that $f_t \neq \emptyset, f_t$ is an ideal of S .

Proof. Let f be a fuzzy ideal of S . Let $t \in [0, 1]$ such that $f_t \neq \emptyset$. Let $x, y \in f_t$. Then $f(x), f(y) \geq t$. Then $f(x + y) \geq \min\{f(x), f(y)\} \geq t$. Next, let $x, y \in S$ and $a \in f_t$. We have $f(xya) \geq \max\{f(x), f(y), f(a)\} \geq f(a) \geq t$. Thus $xya \in f_t$. Similarly, $xay, axy \in f_t$. Therefore f_t is an ideal of S . Conversely, let $x, y, z \in S$ and $t = \min\{f(x), f(y)\}$. Then $f(x), f(y) \geq t$. Thus $x, y \in f_t$. By assumption, $x + y \in f_t$. So $f(x + y) \geq t = \min\{f(x), f(y)\}$. Next, let $s = \max\{f(x), f(y), f(z)\}$. Then $f(x) = s$ or $f(y) = s$ or $f(z) = s$. Thus $x \in f_s$ or $y \in f_s$ or $z \in f_s$. By assumption, $xyz \in f_s$. So $f(xyz) \geq s = \max\{f(x), f(y), f(z)\}$. Therefore f is a fuzzy ideal of S . \square

However, it is not true in general that f is a fuzzy ideal of a ternary semiring S with zero 0, then for any $t \in [0, 1]$ such that $f_t \neq \emptyset, f_t$ is a k-ideal of S . We can see this example.

Example 3.5. Consider the ternary semiring \mathbb{Z}_0^- under usual addition and ternary multiplication. Define a fuzzy subset f on \mathbb{Z}_0^- by

$$f(x) = \begin{cases} 0 & \text{if } x = -1, \\ 0.5 & \text{otherwise.} \end{cases}$$

Then f is a fuzzy ideal of \mathbb{Z}_0^- but $f_{0.5} = \mathbb{Z}_0^- \setminus \{-1\}$ is not a k-ideal of \mathbb{Z}_0^- because $(-1) + (-2) = -3 \in f_{0.5}$ and $-2 \in f_{0.5}$ but $-1 \notin f_{0.5}$.

Theorem 3.6. Let f be a fuzzy subset of a ternary semiring S with zero 0. If for any $t \in [0, 1]$ such that $f_t \neq \emptyset$, f_t is a k -ideal of S , then f is a k -fuzzy ideal of S .

Proof. By Theorem 3.5, f is a fuzzy ideal of S . Next, let $x, y \in S$ such that $f(x+y) = f(0)$ and $f(y) = f(0)$. Then $x+y, y \in f_{f(0)}$. By assumption, $x \in f_{f(0)}$. Hence $f(x) \geq f(0)$. Since f is a fuzzy ideal of S , by Lemma 3.3, $f(x) = f(0)$. Therefore f is a k -fuzzy ideal of S . \square

However, the converse of Theorem 3.6 does not hold. We can see this example.

Example 3.6. Consider the ternary semiring \mathbb{Z}_0^- under usual addition and ternary multiplication. Let f be a fuzzy subset of \mathbb{Z}_0^- defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is even,} \\ 0 & \text{if } x = -1, \\ 0.5 & \text{otherwise.} \end{cases}$$

Then f is a fuzzy ideal of \mathbb{Z}_0^- . Let $x, y \in \mathbb{Z}_0^-$ such that $f(x+y) = f(0)$ and $f(x) = f(0)$. So $f(x+y) = 1$ and $f(y) = 1$. Thus $x+y$ and y are even. Hence x is even, this implies $f(x) = 1 = f(0)$. Therefore f is a k -fuzzy ideal of \mathbb{Z}_0^- . However, $f_{0.5} = \mathbb{Z}_0^- \setminus \{-1\}$ is not a k -ideal of \mathbb{Z}_0^- because $(-1) + (-2) = -3 \in f_{0.5}$ and $-2 \in f_{0.5}$ but $-1 \notin f_{0.5}$.

Definition 3.5. Let S be a ternary semiring with zero 0 and f a fuzzy ideal of S . the k -fuzzy closure \bar{f} of f is defined by

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \notin \overline{f_{f(0)}}, \\ f(0) & \text{if } x \in \overline{f_{f(0)}}. \end{cases}$$

The next theorem holds.

Theorem 3.7. Let S be a ternary semiring with zero 0 and f a fuzzy ideal of S . Then f is a k -fuzzy ideal of S if and only if $f = \bar{f}$.

Proof. Assume f is a k -fuzzy ideal of S and let $x \in \overline{f_{f(0)}}$. Then $\bar{f}(x) = f(0)$. Since $x \in \overline{f_{f(0)}}$, there exist $a, b \in f_{f(0)}$ such that $a+x = b$. Thus $f(a) = f(0)$ and $f(x+a) = f(b) = f(0)$. Then $f(x) = f(0)$. Then $f = \bar{f}$. Conversely, assume $f = \bar{f}$. So $f_{f(0)} = \overline{f_{f(0)}}$, by Theorem 3.1, $f_{f(0)}$ is a k -ideal of S . Let $x, y \in S$ such that $f(x+y) = f(0) = f(y)$. So $x+y, y \in f_{f(0)}$. Then $x \in f_{f(0)}$. So $f(x) = f(0)$. Hence f is a k -fuzzy ideal of S .

Definition 3.6. Let $\varphi : S \rightarrow R$ be a homomorphism of ternary semirings. Let f be a fuzzy subset of R . We define a fuzzy subset $\varphi^{-1}(f)$ of S by

$$\varphi^{-1}(f)(x) = f(\varphi(x)) \text{ for all } x \in S.$$

We call $\varphi^{-1}(f)$ the *preimage* of f under φ .

Theorem 3.8. Let $\varphi : S \rightarrow R$ be an onto homomorphism of ternary semirings. If f be a fuzzy ideal of R , then $\varphi^{-1}(f)$ is a fuzzy ideal of S .

Proof. Let f be a fuzzy ideal of R . Then for any $x, y, z \in S$,

$$\begin{aligned} \varphi^{-1}(f)(x+y) &= f(\varphi(x+y)) \\ &= f(\varphi(x) + \varphi(y)) \\ &\geq \min\{f(\varphi(x)), f(\varphi(y))\} \\ &= \min\{\varphi^{-1}(f)(x), \varphi^{-1}(f)(y)\} \end{aligned}$$

and

$$\begin{aligned} \varphi^{-1}(f)(xyz) &= f(\varphi(xyz)) \\ &= f(\varphi(x)\varphi(y)\varphi(z)) \\ &\geq \max\{f(\varphi(x)), f(\varphi(y)), f(\varphi(z))\} \\ &= \max\{\varphi^{-1}(f)(x), \varphi^{-1}(f)(y), \varphi^{-1}(f)(z)\}. \end{aligned}$$

This shows that $\varphi^{-1}(f)$ is a fuzzy ideal of S . \square

Theorem 3.9. Let S and R be ternary semirings with zero 0 and $\varphi : S \rightarrow R$ an onto homomorphism. Let f be a fuzzy ideal of R . Then f is a k -fuzzy ideal of R if and only if $\varphi^{-1}(f)$ is a k -fuzzy ideal of S .

Proof. Suppose that f is a k -fuzzy ideal of R . Let $x, y \in S$. Assume $\varphi^{-1}(f)(x+y) = \varphi^{-1}(f)(0)$ and $\varphi^{-1}(f)(y) = \varphi^{-1}(f)(0)$. Then $f(\varphi(x+y)) = f(\varphi(0)) = f(0)$ and $f(\varphi(y)) = f(\varphi(0)) = f(0)$. Since f is a k -fuzzy ideal of R , $f(\varphi(x)) = f(0) = f(\varphi(0))$. Thus $\varphi^{-1}(f)(x) = \varphi^{-1}(f)(0)$. Hence $\varphi^{-1}(f)$ is a k -fuzzy ideal of S . Conversely, assume $\varphi^{-1}(f)$ is a k -fuzzy ideal of S . Let $x, y \in R$ such that $f(x+y) = f(0)$ and $f(y) = f(0)$. Since φ is onto, there exist $a, b \in S$ such that $f(a) = x$ and $f(b) = y$. So $f(\varphi(a) + \varphi(b)) = f(\varphi(0))$ and $f(\varphi(b)) = f(\varphi(0))$. Hence $\varphi^{-1}(f)(a+b) = \varphi^{-1}(f)(0)$ and $\varphi^{-1}(f)(b) = \varphi^{-1}(f)(0)$. Since $\varphi^{-1}(f)$ is a k -fuzzy ideal of S , $\varphi^{-1}(f)(a) = \varphi^{-1}(f)(0)$, this implies $f(x) = f(\varphi(a)) = f(\varphi(0)) = f(0)$. Hence f is a k -fuzzy ideal of R . \square

Definition 3.7. Let $\varphi : S \rightarrow R$ be a homomorphism of ternary semirings. Let f be a fuzzy subset of S . We define a fuzzy subset $\varphi(f)$ of R by

$$\varphi(f)(y) = \begin{cases} \sup_{x \in \varphi^{-1}(y)} f(x) & \text{if } \varphi^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We call $\varphi(f)$ the *image* of f under φ .

The following lemma is case $L = [0, 1]$ of Proposition 8 in [10].

Lemma 3.10. ([10]) Let φ be a mapping from a set X to a set Y and f a fuzzy subset of X . Then for every $t \in (0, 1]$,

$$(\varphi(f))_t = \bigcap_{0 < s < t} \varphi(f_{t-s}).$$

Lemma 3.11. The intersection of arbitrary set of ideals of a ternary semiring S is either empty or an ideal of S .

Theorem 3.12. Let $\varphi : S \rightarrow R$ be an onto homomorphism of ternary semirings. If f is a fuzzy ideal of S , then $\varphi(f)$ is a fuzzy ideal of R .

Proof. By Theorem 3.5, it is sufficient to show that each nonempty level subset of $\varphi(f)$ is an ideal of R . Let $t \in [0, 1]$ such that $(\varphi(f))_t \neq \emptyset$. If $t = 0$, then $(\varphi(f))_t = R$. Assume that $t \neq 0$. By Lemma 3.10,

$$(\varphi(f))_t = \bigcap_{0 < s < t} \varphi(f_{t-s}).$$

Then $\varphi(f_{t-s}) \neq \emptyset$ for all $0 < s < t$, and so $f_{t-s} \neq \emptyset$ for all $0 < s < t$. By Theorem 3.5, f_{t-s} is an ideal of S for all

$0 < s < t$. Since φ is an onto homomorphism, $\varphi(f_{t-s})$ is an ideal of R for all $0 < s < t$. By Lemma 3.11, $(\varphi(f))_t = \bigcap_{0 < s < t} \varphi(f_{t-s})$ is an ideal of R . \square

Definition 3.8. Let S and R be any two sets and $\varphi : S \rightarrow R$ be any function. A fuzzy subset f of S is called φ -invariant if $\varphi(x) = \varphi(y)$ implies $f(x) = f(y)$ where $x, y \in S$.

Lemma 3.13. Let S and R be ternary semirings and $\varphi : S \rightarrow R$ a homomorphism. Let f be a φ -invariant fuzzy ideal of S . If $x = \varphi(a)$, then $\varphi(f)(x) = f(a)$.

Proof. If $t \in \varphi^{-1}(x)$, then $\varphi(t) = x = \varphi(a)$. Since f is φ -invariant, $f(t) = f(a)$. This implies

$$\varphi(f)(x) = \sup_{t \in \varphi^{-1}(x)} f(t) = f(a).$$

Hence $\varphi(f)(x) = f(a)$. \square

Theorem 3.14. Let S and R be ternary semirings and $\varphi : S \rightarrow R$ an onto homomorphism. If f is a φ -invariant fuzzy ideal of S , then $\varphi(f)$ is a fuzzy ideal of R .

Proof. Let $x, y, z \in R$. Then there exist $a, b, c \in S$ such that $\varphi(a) = x, \varphi(b) = y$ and $\varphi(c) = z$ and then $x + y = \varphi(a + b)$ and $xyz = \varphi(abc)$. Since f is φ -invariant, by Lemma 3.13, we have

$$\begin{aligned} \varphi(f)(x + y) &= f(a + b) \\ &\geq \min\{f(a), f(b)\} \\ &= \min\{\varphi(f)(x), \varphi(f)(y)\} \end{aligned}$$

and

$$\begin{aligned} \varphi(f)(xyz) &= f(abc) \\ &\geq \max\{f(a), f(b), f(c)\} \\ &= \max\{\varphi(f)(x), \varphi(f)(y), \varphi(f)(z)\}. \end{aligned}$$

Hence $\varphi(f)$ is a fuzzy ideal of R . \square

Theorem 3.15. Let S and R be ternary semirings with zero 0 and $\varphi : S \rightarrow R$ an onto homomorphism. Let f be a φ -invariant fuzzy ideal of S . Then f is a k-fuzzy ideal S if and only if $\varphi(f)$ is a k-fuzzy ideal of R .

Proof. Suppose that f is a k-fuzzy ideal of S and let $x, y \in R$ such that $\varphi(f)(x + y) = \varphi(f)(0)$ and $\varphi(f)(y) = \varphi(f)(0)$. Since φ is onto, there exist $a, b \in S$ such that $\varphi(a) = x$ and $\varphi(b) = y$. By Lemma 3.13, $\varphi(f)(0) = f(0), \varphi(f)(x + y) = f(a + b)$ and $\varphi(f)(y) = f(b)$. Thus $f(a + b) = f(0)$ and $f(b) = f(0)$. Since f is a k-fuzzy ideal of $S, f(a) = f(0)$. By Lemma 3.13, $\varphi(f)(x) = f(a) = f(0) = \varphi(f)(0)$. Hence $\varphi(f)$ is a k-fuzzy ideal of R . Conversely, if $\varphi(f)$ is a k-fuzzy ideal of R , then for any $x \in S$,

$$\varphi^{-1}(\varphi(f))(x) = \varphi(f)(\varphi(x)) = f(x).$$

So $\varphi^{-1}(\varphi(f)) = f$. Since $\varphi(f)$ is a k-fuzzy ideal of R , by Theorem 3.9, $f = \varphi^{-1}(\varphi(f))$ is a k-fuzzy ideal of S . \square

Next, we define fuzzy k-ideals of ternary semirings analogous to fuzzy k-ideals of semirings.

Definition 3.9. A fuzzy ideal f of a ternary semiring S is said to be a fuzzy k-ideal of S if

$$f(x) \geq \min\{f(x + y), f(y)\}$$

for all $x, y \in S$.

Example 3.7. Let f be a fuzzy subset of a ternary semiring \mathbb{Z}_0^- under the usual addition and ternary multiplication defined by

$$f(x) = \begin{cases} 0 & \text{if } x = -1, \\ 0.5 & \text{otherwise.} \end{cases}$$

Then f is a fuzzy ideal of \mathbb{Z}_0^- but not a fuzzy k-ideal of \mathbb{Z}_0^- because set $x = -1$ and $y = -2$, we have $f(x) = 0 < 0.5 = \min\{f(x + y), f(y)\}$.

Example 3.8. let f be a fuzzy subset of a ternary semiring \mathbb{Z}_0^- under usual addition and ternary multiplication defined by

$$f(x) = \begin{cases} 0.3 & \text{if } x \text{ is odd,} \\ 0.5 & \text{if } x \text{ is even.} \end{cases}$$

It is easy to show that f is a fuzzy k-ideal of \mathbb{Z}_0^- .

Lemma 3.16. Let S be a ternary semiring and f a fuzzy ideal of S . Then f is a fuzzy k-ideal of S if and only if for any $t \in [0, 1]$ such that $f_t \neq \emptyset, f_t$ is a k-ideal of S .

Proof. By Theorem 3.5, f_t is an ideal of S . Let $x, y \in f_t$ and assume $x + y, y \in f_t$. Then $f(x + y), f(y) \geq t$. Since f is a fuzzy k-ideal of $S, f(x) \geq \min\{f(x + y), f(y)\} \geq t$. So $x \in f_t$. Therefore f_t is a k-ideal of S . Conversely, assume for any $t \in [0, 1]$ such that $f_t \neq \emptyset, f_t$ is a k-ideal of S . By Theorem 3.5, f is a fuzzy ideal of S . Next, let $x, y \in S$. Set $t = \min\{f(x + y), f(y)\}$. Then $f(x + y), f(y) \geq t$. So $x + y, y \in f_t$. By assumption, f_t is a k-ideal of S , this implies $x \in f_t$. Hence $f(x) \geq t = \min\{f(x + y), f(y)\}$. Therefore f is a fuzzy k-ideal of S . \square

Theorem 3.17. Let S be a ternary semiring with zero 0 and f a fuzzy ideal of S . If f is a fuzzy k-ideal of S , then f is a k-fuzzy ideal of S .

Proof. Let $x, y \in S$ such that $f(x + y) = f(0)$ and $f(y) = f(0)$. Set $t = f(0)$. So $x + y, y \in f_t$. By Lemma 3.16, the level subset f_t is a k-ideal of S . So $x \in f_t$. This implies $f(x) \geq t = f(0)$. By Lemma 3.3, $f(x) = f(0)$. \square

However, the converse of Theorem 3.17 does not hold. We can see this example.

Example 3.9. Consider the ternary semiring \mathbb{Z}_0^- under usual addition and ternary multiplication. Define a fuzzy subset f on \mathbb{Z}_0^- by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is even,} \\ 0 & \text{if } x = -1, \\ 0.5 & \text{otherwise.} \end{cases}$$

By Example 3.6, we know that f is a k-fuzzy ideal of \mathbb{Z}_0^- . However, f is not a fuzzy k-ideal of \mathbb{Z}_0^- because $f((-1) + (-2)) = f(-3) = 0.5$ and $f(-2) = 1$ but $f(-1) = 0 < 0.5 = \min\{f((-1) + (-2)), f(-2)\}$.

Definition 3.10. Let f and g be fuzzy subset of a non-empty subset S . A fuzzy subset $f \cap g$ of S is defined by $(f \cap g)(x) = \min\{f(x), g(x)\}$ for all $x \in S$.

Lemma 3.18. Let f and g be fuzzy subset of a ternary semiring S . If f and g are fuzzy ideals of S , then $f \cap g$ is a fuzzy ideal of S .

Proof. Let $x, y, z \in S$. We have

$$\begin{aligned}(f \cap g)(x + y) &= \min\{f(x + y), g(x + y)\} \\ &\geq \min\{f(x), f(y), g(x), g(y)\} \\ &= \min\{(f \cap g)(x), (f \cap g)(y)\}\end{aligned}$$

and

$$\begin{aligned}(f \cap g)(xyz) &= \min\{f(xyz), g(xyz)\} \\ &\geq \min\{\max\{f(x), f(y), f(z)\}, \max\{g(x), g(y), g(z)\}\} \\ &\geq \max\{(f \cap g)(x), (f \cap g)(y), (f \cap g)(z)\}.\end{aligned}$$

Hence $f \cap g$ is a fuzzy ideal of S . \square

Theorem 3.19. Let f and g be fuzzy subset of a ternary semiring S . If f and g are fuzzy k-ideals of S , then $f \cap g$ is a fuzzy k-ideal of S .

Proof. By Lemma 3.18, $f \cap g$ is a fuzzy ideal of S . Let $x, y \in S$. We have

$$\begin{aligned}(f \cap g)(x) &\geq \min\{f(x), g(x)\} \\ &\geq \min\{f(x + y), f(y), g(x + y), g(y)\} \\ &= \min\{(f \cap g)(x + y), (f \cap g)(y)\}.\end{aligned}$$

Hence $f \cap g$ is a fuzzy k-ideal of S . \square

Let f and g be k-fuzzy ideals of a ternary semiring S . In general, a fuzzy ideal $f \cap g$ need not be a k-fuzzy ideal of S . See this example.

Example 3.10. Consider the ternary semiring \mathbb{Z}_0^- under usual addition and ternary multiplication. Let f and g be fuzzy subsets on \mathbb{Z}_0^- by

$$f(x) = \begin{cases} 0.3 & \text{if } x = 0, \\ 0.1 & \text{if } x = -1, \\ 0.2 & \text{otherwise} \end{cases}$$

and $g(x) = 0.2$ for all $x \in \mathbb{Z}_0^-$. It is easy to verify that f and g are k-fuzzy ideal of \mathbb{Z}_0^- . We have

$$(f \cap g)(x) = \begin{cases} 0.1 & \text{if } x = -1, \\ 0.2 & \text{otherwise.} \end{cases}$$

Set $x = -1$ and $y = -2$. We have $(f \cap g)(x + y) = 0.2 = (f \cap g)(0)$ and $(f \cap g)(y) = 0.2 = (f \cap g)(0)$ but $(f \cap g)(x) = 0.1 \neq 0.2 = (f \cap g)(0)$. Thus $f \cap g$ is not k-fuzzy ideal of \mathbb{Z}_0^- .

ACKNOWLEDGMENT

This research is (partially) supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

REFERENCES

- [1] M. Akram and W. A. Dudek, *Intuitionistic fuzzy left k-ideals of semirings*, Soft Computing **12** (2008), 881-890.
- [2] T. K. Dutta and B. K. Biswas, *Fuzzy k-ideals of semirings*, Bulletin of the Calcutta Mathematical Society **87** (1995), 91-96.
- [3] T. K. Dutta and S. Kar, *On regular ternary semirings*, Advances in Algebra, Proceedings of the ICM Satellite Conference in Algebra and Related Topics, World Scientific, 343-355, 2003.
- [4] T. K. Dutta and S. Kar, *On the Jacoson radical of a ternary semiring*, Southeast Asian Bulletin of Mathematics **28** (2004), 1-13.
- [5] T. K. Dutta and S. Kar, *A note on the Jacoson radical of a ternary semiring*, Southeast Asian Bulletin of Mathematics **29** (2005), 321-331.
- [6] T. K. Dutta and S. Kar, *Two types Jacoson radicals of ternary semirings*, Southeast Asian Bulletin of Mathematics **29** (2005), 677-687.
- [7] T. K. Dutta and S. Kar, *A note on regular ternary semirings*, Kyungpook Mathematical Journal **46** (2006), 357-365.
- [8] S. Ghosh, *Fuzzy k-ideals of semirings*, Fuzzy Sets and Systems **95** (1998), 103-108.
- [9] H. Hedayati, *Generalized fuzzy k-ideals of semirings with interval-valued membership functions*, Bulletin of the Malaysian Mathematical Sciences Society **32** (2009), 409-424.
- [10] Y. B. Jun, J. Neggers and H. S. Kim, *On L-fuzzy ideals in semirings I*, Czechoslovak Mathematical Journal **48** (1998), 669-675.
- [11] S. Kar, *On quasi-ideals and bi-ideals in ternary semirings*, International Journal of Mathematics and Mathematical Sciences **2005** (2005), 3015-3023.
- [12] J. Kavikumar and A. B. Khamis, *Fuzzy ideals and fuzzy quasi-ideals in ternary semirings*, IAENG International Journal of Applied Mathematics **37** (2007), 102-106.
- [13] J. Kavikumar, A. B. Khamis and Y. B. Jun, *Fuzzy bi-ideals in ternary semirings*, International Journal of Computational and Mathematical sciences **3** (2009), 164-168.
- [14] C. B. Kim, M. A. Park, *k-Fuzzy ideals in semirings*, Fuzzy Sets and Systems **81** (1996), 281-286.
- [15] D. H. Lehmer, *A ternary analogue of abelian groups*, American Journal of Mathematics **54** (1932), 329-388.
- [16] W. G. Lister, *Ternary rings*, Transaction of American Mathematical Society **154** (1971), 37-55.
- [17] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338-353.

Sathinee Malee was born in Yala, Thailand, in 1985. She finished her B.Sc. from Prince of Songkla University in 2007. Now she is continuing her graduation in Master degree at same university. Both of Bachelor and Master level she got a Science Achievement Scholarship of Thailand to support her education.

Ronnason Chinram was born in Ranong, Thailand, in 1975. He received his M.Sc and Ph.D. from Chulalongkorn University, Thailand. Since 1997, he has been with Prince of Songkla University, Thailand where now he is an Assistant Professor in Mathematics. His research interests focus on semigroup theory and algebraic systems.