Counting the number of distinct fuzzy subgroups some of the dihedral groups

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Abstract—In this paper, by using of an equivalence relation on fuzzy subgroup, we determine the number of distinct fuzzy subgroups some of the dihedral groups.

Keywords—Fuzzy subgroups, Dihedral group, Equivalence relation.

I. Introduction

In his classic paper [15] of 1965, Zadeh introduced the notion of fuzzy sets and fuzzy set operations. In an analogous application with groups, Rosenfeld [12] formulated the elements of a theory of fuzzy groups. One of the most important problems of fuzzy group theory is to classify the fuzzy subgroups of a finite group. This topic has enjoyed a rapid evolution in the last years. Many papers have treated the particular case of finite cyclic groups. Thus, in [8] the number of distinct fuzzy subgroups of a finite cyclic group of square-free order is determined, while [9]-[11], [13] deal with this number for cyclic groups of order $p^n q^m$ (p, q primes). In the present paper we establish the recurrence relation verified by the number of distinct fuzzy subgroups some of the dihedral groups. First of all, we present some basic notions and results of fuzzy subgroup theory (for more details, see [3], [6]).

II. Preliminaries

The dihedral group of order $2n$, for $n \geq 2$, denoted by D_{2n}. A fuzzy subset of a set X is mapping $\mu :\to [0,1]$. Fuzzy subset μ of a group G is called a fuzzy subgroup of G if
(G_1) $\mu(xy) \geq \mu(x) \wedge \mu(y) \forall x, y \in G$;
(G_2) $\mu(x^{-1}) \geq \mu(x) \forall x \in G$.
The set of all fuzzy subgroup of a group G denoted by $F(G)$.

Definition 2.1: Let G be a group and $\mu \in F(G)$. The set $\{x \in G | \mu(x) > 0\}$ is called support of μ and denoted by $\text{supp}\mu$.
Let G be a group and $\mu \in F(G)$. We shall write $\text{Im}\mu$ for the image set of μ and F_μ for the family $\{\mu_t | t \in \text{Im}\mu\}$.

Definition 2.2: Let G be a group, and $\mu, \nu \in F(G)$. In [5] defined three equivalence relation as follow respectively:
(i) We say that μ is equivalence ν, written as $\mu \approx \nu$ if $F_\mu = F_\nu$.
(ii) We say that μ is equivalent to ν, written as $\mu \sim \nu$ if

1) $\mu(x) > \mu(y) \Leftrightarrow \nu(x) > \nu(y)$, for all $x, y \in G$.
2) $\mu(x) = 0 \Leftrightarrow \nu(x) = 0$, for all $x \in G$.

(iii) We say that μ is equivalence ν, written as $\mu \simeq_t \nu$ if there exists an isomorphism f from $\text{supp}\mu$ to $\text{supp}\nu$ such that for all $x, y \in \text{supp}\mu$,

$$\mu(x) > \mu(y) \Leftrightarrow \nu(f(x)) > \nu(f(y))$$

Definition 2.3: Let G be a group and $\mu, \nu \in F(G)$. We say that μ is equivalence ν, written as $\mu \sim_t \nu$, if and only if $F_\mu = F_\nu$ and $\text{supp}\mu = \text{supp}\nu$.

III. Counting of the distinct fuzzy subgroups of the dihedral group D_{2p^n}

Let $F_1(G)$ be the set of all fuzzy subgroups μ of G such that $\mu(e) = 1$ and let $\sim_R$ be an equivalence relation on $F_1(G)$. We denote the set $\{\nu \in F_1(G) | \nu \sim_R \mu\}$ by $\frac{\mu}{\sim_R}$ and the set $\{\frac{\mu}{\sim_R} | \mu \in F_1(G)\}$ by $\frac{F_1(G)}{\sim_R}$.

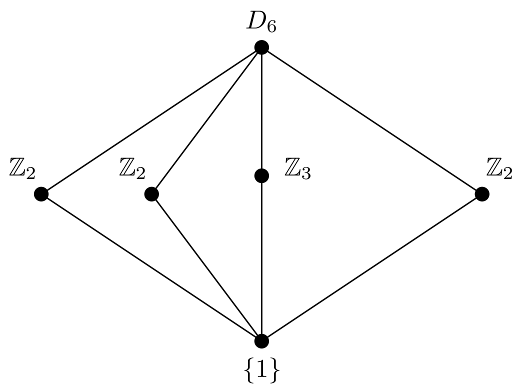
Theorem 3.1: Let G be a finite group. then $|\frac{F_1(G)}{\approx}| = \frac{|\frac{F_1(G)}{\sim}+1|}{2}$
The number of the equivalence classes $\sim$ on $F_1(G)$ will be denoted by $r^\star_G$.

Theorem 3.2: Let G be a finite group and H be a subgroup of G. Then the number of distinct fuzzy subgroups of G such that their support is exactly equal to H is $\frac{r^\star_H+1}{2}$.

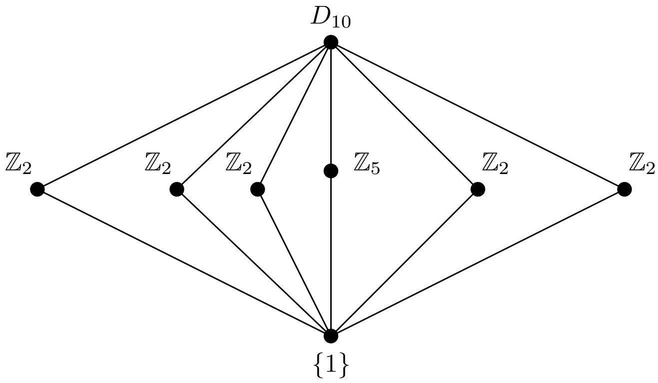
Theorem 3.3: Let G be a finite group and H be a subgroup of G. Then the number of distinct fuzzy subgroups of G such that their support is exactly a subgroup of H is $\frac{r^\star_H-1}{2}$.

Proposition 3.4: [5]. Let $n \in \mathbb{N}$. Then there are $2^{n+1}-1$ distinct equivalence classes of fuzzy subgroups of $\mathbb{Z}_{p^n}$.

Example 3.5: Let G be the dihedral group of order 6. Then $r^\star_G = 19$.



Example 3.6: Let G be the dihedral group of order 10. Then $r^\star_G = 27$.



Theorem 3.7: Suppose that p is a prime and $p \geq 3$. If G is the dihedral group of order $2p$, Then $r^\star_G = 4p + 7$.

Proposition 3.8: Suppose that p is a prime and $p \geq 3$. If G is the dihedral group of order $2p^2$, Then $r^\star_G = 8p^2 + 8p + 15$.

Proposition 3.9: Suppose that p is a prime and $p \geq 3$. If G is the dihedral group of order $2p^3$, Then $r^\star_G = 16p^3 + 16p^2 + 16p + 31$.

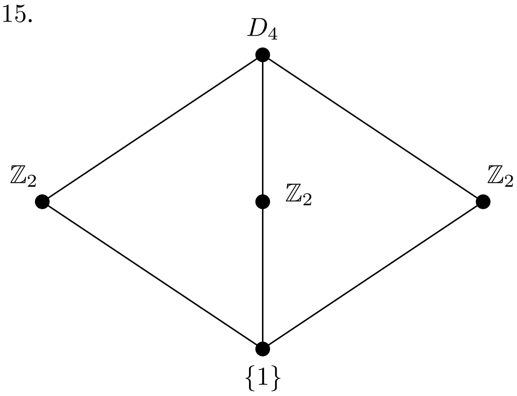
Theorem 3.10: Suppose that p is a prime and $p \geq 3$. If G is the dihedral group of order $2p^n$, Then $r^\star_G = \sum_{i=1}^{n-1} p^i r^\star_{(D_{2p^{n-i}})} + \frac{p^n - p}{p-1} + 2^{n+2} + 4p^n - 1$.

Corollary 3.11: Suppose that p is a prime and $p \geq 3$, Then $r^\star_{(D_{2p^n})} = \frac{2^{n+1}p(p^n-1)}{p-1} + 2^{n+2} - 1$.

IV. Counting of the distinct fuzzy subgroups of the dihedral group D_{2^n}

Now in this section, we determine the number of distinct fuzzy subgroups of the dihedral group D_{2^n}. First notice some examples as follows:

Example 4.1: Let G be the dihedral group of order 4, then $r^\star_G = 15$.



Example 4.2: Let G be the dihedral group of order 8, then $r^\star_G = 91$.

Example 4.3: Let G be the dihedral group of order 16, then $r^\star_G = 547$.

Example 4.4: Let G be the dihedral group of order 32, then $r^\star_G = 3283$.

Example 4.5: Let G be the dihedral group of order 64, then $r^\star_G = 19699$.

Theorem 4.6: Let G be the dihedral group of order 2^n. If $n \geq 3$, then $r^\star_G = 2^{n+2} + 4(\sum_{i=1}^{n-2} 2^{i-1} r^\star_{(D_{2^{n-i}})}) - 1$.

Corollary 4.7: If $n \geq 3$, then $r^\star_{(D_{2^n})} = 6r^\star_{(D_{2^{n-1}})} + 1$. Thus $r^\star_{(D_{2^n})} = \frac{76 \times 6^{n-2} - 1}{5}$.

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