On some subspaces of Entire sequence space of Fuzzy Numbers

T. Balasubramanian and A. Pandiarani

Abstract—In this paper we introduce some subspaces of fuzzy entire sequence space. Some general properties of these sequence spaces are discussed. Also some inclusion relation involving the spaces are obtained.

Mathematics Subject Classification: 40A05, 40D25.
Keywords—Fuzzy Numbers, Entire sequences, completeness, Fuzzy entire sequences

I. Introduction

The concepts of fuzzy set theory was introduced by Zadeh [1]. Later on sequence of fuzzy numbers have been discussed by Matloka [2] and developed by Mursaleen [3] Nanda [4] and Savas [5], Tripathy and Dutta [6] and many others. The sequence space Γ of entire sequences was introduced by Ganapathy Iyer [9]. The space $\Gamma(F)$ of fuzzy entire sequences space was introduced by Kavikumar, Azme Bin Khamis and Kandasamy [8]. Also Orlicz space of Entire sequence of fuzzy numbers was introduced by Subramanian and Metin Basarir [9]. In this article we introduce some subspaces of fuzzy entire sequence space and some of their properties are discussed

II. Preliminaries and Definitions

We begin with giving some required definitions and propositions and lemmas. A fuzzy number is a fuzzy set on the real axis i.e. a mapping $u : \mathbb{R} \to [0,1]$ which satisfies the following four conditions.

(i) u is normal i.e. there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.
(ii) u is fuzzy convex i.e. $u[\lambda x + (1-\lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0,1]$.
(iii) u is upper semi-continuous.
(iv) The set $[u]_0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact (cf. Zadeh [1]) where $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$ in the usual topology of $\mathbb{R}$. We denote the set of all fuzzy numbers on $\mathbb{R}$ by E' and called it as the space of fuzzy numbers. α-level set $[u]_\alpha$ of $u \in E'$ is defined by $[u]_\alpha = \begin{cases} \{t \in \mathbb{R} : u(t) \geq \lambda\}, & (0 < \lambda \leq 1) \\ \overline{\{t \in \mathbb{R} : u(t) > \lambda\}}, & (\lambda = 0). \end{cases}$
The set $[u]_\alpha$ is a closed bounded and non-empty interval for each $\alpha \in [0,1]$ which is defined by $[u]_\alpha = [u^-(\alpha), v^-(\alpha)]$.

T. Balasubramanian, Department of Mathematics, Kamaraj College, Tuticorin, Tamilnadu, India. Email:satbalu@yahoo.com
A. Pandiarani, Department of Mathematics, G. Venkataswamy Naidu College, Kovilpatti, Tamilnadu, India. Email: raniseelan_92@yahoo.co.in.

$\mathbb{R}$ can be embedded in E' since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\bar{r}$ defined by $\bar{r} = \begin{cases} 1, & (x = r) \\ 0, & (x \neq r). \end{cases}$
Let $u, v, w \in E'$ and $k \in \mathbb{R}$. Then the operations addition, scalar multiplication and product and division defined on E' by

$$\begin{aligned} u + v = w &\Leftrightarrow [w]\alpha = [u]_\alpha + [v]_\alpha \quad \text{for all} \quad \alpha \in [0,1] \\ &\Leftrightarrow w^-(\alpha) = [u^-(\alpha), v^-(\alpha)] \quad \text{and} \\ w^+ &= [u^+(\alpha), v^+(\alpha)] \text{ for all } \alpha \in [0,1] \\ [ku]_\alpha &= k[u]_\alpha \quad \text{for all} \quad \alpha \in [0,1] \\ \text{and} \quad uv = w &\Leftrightarrow [w]_\alpha = [u]_\alpha [v]_\alpha \quad \text{for all} \quad \alpha \in [0,1] \end{aligned}$$

where it is immediate that

$$\begin{aligned} w^-(\alpha) &= \min\{u^-(\alpha), v^-(\alpha), u^-(\alpha), v^+(\alpha), \\ &\qquad u^+(\alpha), v^-(\alpha), u^+(\alpha), v^+(\alpha)\} \\ \text{and} \quad w^+(\alpha) &= \max\{u^-(\alpha), v^-(\alpha), u^-(\alpha), v^+(\alpha), \\ &\qquad u^+(\alpha), v^-(\alpha), u^+(\alpha), v^+(\alpha)\} \\ \frac{u}{v} &= w = [w]_\alpha = [u]_\alpha / [v]_\alpha \quad \text{for all} \quad \alpha \in [0,1] \\ &= [u^-(\alpha), v^+(\alpha)] \cdot \left[\frac{1}{v^-(\alpha)}, \frac{1}{v^+(\alpha)} \right] \\ &= \left[\min\left\{ \frac{[u]^-(\alpha)}{[v]^+(\alpha)}, \frac{u^-(\alpha)}{v^-(\alpha)}, \frac{u^+(\alpha)}{v^+(\alpha)}, \frac{u^+(\alpha)}{v^-(\alpha)} \right\} \right] \end{aligned}$$

Let W be the set of all closed and bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$ i.e., $A = [\underline{A}, \overline{A}]$.
Define the relation d on W by $d(A, B) = \max\{|\overline{A} - \overline{B}|\}$. Then it can easily be observed that d is a metric on w (cf. Diamond and Kloeden [10]) and (W, d) is a complete metric space. Now we can define the metric D on E' by means of a Hausdroff metric d as

$$\begin{aligned} D(u, v) &= \sup_{\alpha \in [0,1]} d([u]_\alpha, [v]_\alpha) \\ &= \sup_{\alpha \in [0,1]} \max\{|u^-(\alpha) - v^-(\alpha)|, |u^+(\alpha) - v^+(\alpha)|\} \end{aligned}$$

One can extend the natural order relation on the real line to intervals as follows:

$$A \leq B \quad \text{if and only if} \quad \underline{A} \leq B \quad \text{and} \quad \overline{A} \leq \overline{B}$$

The partial order relation on E' is defined as follows. $u \leq v$ if and only if $[u]_\alpha \leq [v]_\alpha$ if and only if $u^-(\alpha) \leq v^-(\alpha)$ and $u^+(\alpha) \leq v^+(\alpha)$.

An absolute value $|u|$ of a fuzzy number u is defined by

$$|u|(t) = \begin{cases} \max\{u(t), u(-t)\}, & (t \geq 0) \\ 0, & (t > 0) \end{cases}$$

α-level set $[|u|]_\alpha$ of the absolute value of $u \in E'$ is in the form
$[|u|]_\alpha = [u^-(\alpha), v^+(\alpha)]$ where

$$|u|^-(\alpha) = \max\{0, u^-(\alpha), -u^+(\alpha)\}$$
$$|u|^+(\alpha) = \max\{|u^-(\alpha)|, |u^+(\alpha)|\}$$

Definition II.1. *A sequence $u = (u_k)$ of fuzzy numbers is a function u from the set $\mathbb{N}$ into E'.*
The fuzzy number u_k denotes the value of the function at $k \in \mathbb{N}$ and is called the k-th term of the sequence. The set of all fuzzy sequences is denoted by $w(F)$.

Definition II.2. *A sequence $u = (u_k) \in w(F)$ is called convergent with limit $\ell \in E'$ if and only if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $D(u_k, u) < \varepsilon$ for all $k \geq n_0$.*

Definition II.3. *A sequence $u = (u_k) \in w(F)$ is called bounded if and only if the set of fuzzy numbers consisting of the terms of the sequence (u_k) is a bounded set. That is, a sequence $(u_k) \in w(F)$ is said to be bounded if and only if there exist two fuzzy numbers m and M such that $m \leq u_k \leq M$ for all $k \in \mathbb{N}$. In this article we define some sub spaces $\Gamma(F, \lambda)$, $\chi(F)$ and $\Gamma(F, 1, d)$ of fuzzy entire sequences space.*

III. Main Results

For each fixed k, define the fuzzy metric

$$D(u_k, v_k) = \sup_{\alpha \in [0,1]} \max\{|u_k^-(\alpha) - v_k^-(\alpha)|^{1/k}, |u_k^+(\alpha) - v_k^+(\alpha)|^{1/k}\}$$

Clearly (E', D) is a complete metric space.

We define the space of all entire sequences of fuzzy numbers by

$$\Gamma(F) = \{u = (u(k)) \in w(F) : \lim_{k \to \infty} D(u_k, \overline{0}) = 0\}$$

Theorem III.1. *$\Gamma(F)$ is a complete metric space with respect to the metric*

$$d(u, v) = \sup_k D(u_k, v_k)$$

Proof: Let $\{u^i\}$ be a Cauchy sequence of fuzzy numbers in $\Gamma(F)$. Then for given $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$d(u^i, u^j) = \sup_k D[u_k^{(i)}, v_k^{(j)}] < \varepsilon \quad \text{for all } i, j \geq n_0 \tag{1}$$

Since this is true for all k, we have

$$D(u_k^{(i)}, v_k^{(j)}) < \varepsilon \quad \text{for all } i, j \geq n_0 \tag{2}$$

This leads to the fact that $\{u_k^{(i)}\}$ is a Cauchy sequence of fuzzy number in E'. Since (E', D) is a complete metric space, $\{u_k^{(i)}\} \to u_k$ as $i \to \infty$.
Therefore, $D(u_k^{(i)}, u_k) < \varepsilon$, since this is true for all k, $\sup_k D(u_k^{(i)}, u_k) < \varepsilon$ implies that $u^{(i)} \to u$ in $\Gamma(F)$.
It is easy to see that $u \in \Gamma(F)$.
Hence $\Gamma(F)$ is complete. ■

Now we proceed to define some subspaces of $\Gamma(F)$.

Definition III.2. *Let $\lambda = (\lambda_k)$ denote a fixed sequence of fuzzy numbers such that $\lambda_k \neq 0$ for all k. We define the sequence space $\Gamma(F, \lambda)$ as follows:*

$$\Gamma(F, \lambda) = \{u \in \Gamma(F) : \lambda u \in \Gamma(F)\}.$$

Theorem III.3. *$\Gamma(F, \lambda) = \Gamma(F)$ if and only if $\limsup\{D(\lambda_k, \overline{0})\} < \infty$.*

Proof: Suppose $\Gamma(F, \lambda) = \Gamma(F)$.
Let $u \in \Gamma(F, \lambda)$. Then $\lambda u \in \Gamma(F)$. Therefore for given $\varepsilon > 0$ there exist n_0 such that $D(\lambda_k u_k, \overline{0}) < \varepsilon$ for all $k \geq n_0$. Suppose $\limsup\{D(\lambda_k, \overline{0})\} = \infty$. Then there exist a subsequence $\{n_k\}$ such that $D(\lambda_{n_k}, \overline{0}) > M$ for some $M > 0$.
Therefore $\sup_{\alpha \in [0,1]} \max\left\{|\lambda_{n_k}^+(\alpha)|^{1/k}, |\lambda_{n_k}^-(\alpha)|^{1/k}\right\} > M$.
This implies that $|\lambda_{n_k}^+(\alpha)|^{1/k} > M$ and $|\lambda_{n_k}^-(\alpha)|^{1/k} > M$.
Now, define a sequence of fuzzy numbers by

$$u_{n_k} = \begin{cases} \overline{1}, & \text{if } n = k, \\ 0, & \text{if } n \neq k. \end{cases}$$

Then $u_{n_k}^-(\alpha) = 0$ and $u_{n_k}^+(\alpha) = 1$. Clearly $(u_{n_k} \in\in \Gamma(F)$.
But $|u_{n_k}^+(\alpha)\lambda_{n_k}^-(\alpha)|^{1/k} > M$ and $|u_{n_k}^+(\alpha)\lambda_{n_k}^+(\alpha)|^{1/k} > M$, which contradicts the fact that $D(\lambda_k u_k, \overline{0}) < \varepsilon$.
Conversely, Suppose $\limsup\{D(\lambda_k, \overline{0}\} < \infty$. Then there exist $M > 0$ such that $D(\lambda_k, \overline{0}) < M$ for all k.
Obviously $\Gamma(F, \lambda) \subseteq \Gamma(F)$. Let $u \in \Gamma(F)$. Then $D(u_k, \overline{0}) < \varepsilon/M$. Now, $D(\lambda_K u_k, \overline{0}) \leq D(\lambda_k, \overline{0}) \leq D(u_k, \overline{0}) < \varepsilon$ (see cf. Talo [11]).
Hence $\lambda u \in \Gamma(F)$. From this we get $\Gamma(F) \subseteq \Gamma(F, \lambda)$.
Consequently, $\Gamma(F) = \Gamma(F, \lambda)$. This completes the proof ■

Theorem III.4. *If $\lambda = (\lambda_k)$ and $\mu = (\mu_k)$ are any two fixed sequences of fuzzy numbers and if $\{D(\gamma_k, \overline{0})\} < M$ for all k and for some $M > 0$, where $\gamma_k = \frac{\mu_k}{\lambda_k}$ then $\Gamma(F, \lambda) \subset \Gamma(F, \mu)$.*

Proof: Suppose $\{D(\gamma_k, \overline{0})\} < M$ for some $M > 0$.
Then $\sup_{\alpha \in [0,1]} \max\left\{|\gamma_k^+(\alpha)|^{1/k}, |\gamma_k^-(\alpha)|^{1/k}\right\} < M$.
This implies that $\left|\left(\frac{\mu_k}{\lambda_k}\right)^-(\alpha)\right|^{1/k} < M$ and $\left|\left(\frac{\mu_k}{\lambda_k}\right)^+(\alpha)\right|^{1/k} < M$. From this we get

$$|\mu_k^-| < M\left|\lambda_k^-(\alpha)\right|, \quad |\mu_k^+| < M\left|\lambda_k^-(\alpha)\right| \tag{3a}$$
$$\text{and} \quad |\mu_k^-| < M\left|\lambda_k^+(\alpha)\right|, \quad |\mu_k^+| < M\left|\lambda_k^+(\alpha)\right| \tag{3b}$$

Let $u \in \Gamma(F, \lambda)$. Then $\lambda u \in \Gamma(F)$. Therefore for given $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $D(\lambda_k u_k, \overline{0}) < \varepsilon/M$ for all $k \geq n_0$.
This implies that
$\sup_{\alpha\in[0,1]} \max\{|(\lambda_k u_k)^-(\alpha)|^{1/k}, |(\lambda_k u_k)^+(\alpha)|^{1/k}\} < \varepsilon/M$.
This means that $|(\lambda_k u_k)^-(\alpha)|^{1/k} < \varepsilon/M$ and $|(\lambda_k u_k)^+(\alpha)|^{1/k} < \varepsilon/M$. From this we get

$$|\lambda_k^-(\alpha)u_k^-(\alpha)| < \varepsilon/M, \quad |\lambda_k^+(\alpha)u_k^+(\alpha)| < \varepsilon/M \tag{4}$$

Using (3) and (4) we get

$$|\mu_k^-(\alpha)u_k^-(\alpha)| = |\mu_k^-(\alpha)|\,|u_k^-(\alpha)| < M,$$
$$|\lambda_k^-(\alpha)u_k^-(\alpha)| < M \cdot \frac{\varepsilon}{M} = \varepsilon$$

Similarly we have $|\mu_k^-(\alpha)u_k^+(\alpha)| < \varepsilon$
and $|\mu_k^+(\alpha)u_k^-(\alpha)| < \varepsilon$, $|\mu_k^+(\alpha)u_k^+(\alpha)| < \varepsilon$. Hence $\sup_{\alpha\in[0,1]} \max\{|(\mu_k u_k)^-(\alpha)|^{1/k}, \ |(\mu_k u_k)^+(\alpha)|^{1/k}\} < \varepsilon$.
This implies that $D(\mu_k u_k, \overline{0}) < \varepsilon$. Thus $u \in \Gamma(F, \mu)$. ■

Remark: The condition stated in Theorem III.5 is not necessary. Let us, now define a sequence $(\lambda_k) = \left(\frac{1}{k!}\right)$, where $k \in E'$ and $\mu_k = (\overline{1})$ for all k.
Then $\{D(\gamma_k, \overline{0})\}$ and $\{D(\mu_k, \overline{0})\}$ are bounded sequences.
Thus by Theorem III.4, $\Gamma(F, \lambda) = \Gamma(F, \mu) = \Gamma(F)$.
Therefore $\Gamma(F, \lambda) \subseteq \Gamma(F, \mu)$.
But $\left\{D\left(\frac{\mu_k}{\lambda_k}, \overline{0}\right)\right\}$ is unbounded. □

$\Gamma(F, \lambda)$ is endowed with two topologies, one is the metric topology inherited from $\Gamma(F)$, its metric being

$$d(u,v) = \sup_k \Big\{ \sup_{\alpha\in[0,1]} \max\{|u_k^-(\alpha) - v_k^-(\alpha)|^{1/k}, \\ |u_k^+(\alpha) - v_k^+(\alpha)|^{1/k}\Big\}$$

where $u, v \in \Gamma(F, \lambda)$. The other is the metric topology d_λ given by
$d_\lambda(u,v) = \sup_k D_\lambda(u_k, v_k)$, where

$$D_\lambda(u_k, v_k) = \sup_{\alpha\in[0,1]} \max\big\{|\lambda_k^-(\alpha)|^{1/k}|u_k^-(\alpha) - v_l^-(\alpha)|^{1/k}, \\ |\lambda_k^-(\alpha)|^{1/k}|u_k^+(\alpha) - v_l^+(\alpha)|^{1/k}\big\}$$

and E' is complete with respect to D_λ.

Theorem III.5. *If* $\limsup\{D(\lambda_k, \overline{0})\} < \infty$ *then* d *is finer than* d_λ.

Proof: To prove the result, it is enough to prove that if $\{u_k\}$ is a sequences of fuzzy numbers converging to u in $[\Gamma(F, \lambda), d]$ then the sequence converges to u in $[\Gamma(F, \lambda), d_\lambda]$. Consider the identity mapping I from $(\Gamma(F, \lambda), d)$ to $(\Gamma(F, \lambda), d_\lambda)$ defined by $u \to u$. Take $u = \overline{0}$, where $\overline{0}$ is the zero element of $\Gamma(F)$. Since (u_k) converges to $u = \overline{0}$ in $(\Gamma(F, \lambda), d)$, given $\varepsilon > 0$ there exists a positive integer n_0 such that $d(u, \overline{0}) = \sup D(u_k, \overline{0}) < \varepsilon$ for all $k \geq n_0$.

Let $U = \limsup\{D(\lambda_k, \overline{0})\}$

$$d_\lambda(u, \overline{0}) = \sup_k \Big\{ \sup_{\alpha\in[0,1]} \max\big\{|\lambda_k^-(\alpha)|^{1/k}|u_k^-(\alpha)|^{1/k}, \\ |\lambda_k^+(\alpha)|^{1/k}|u_k^+(\alpha)|^{1/k}\big\}\Big\}$$
$$\leq U \sup_k \Big\{ \sup_{\alpha\in[0,1]} \max\big\{|u_k^-(\alpha)|^{1/k}, \, |u_k^+(\alpha)|^{1/k}\big\}\Big\}$$
$$\leq U \sup_k D(u_k, \overline{0}) < U\varepsilon$$

Hence (u_k) converges to $\overline{0}$ in $\big(\Gamma(F, \lambda), d_\lambda\big)$. ■

Theorem III.6. $\big(\Gamma(F, \lambda), d_\lambda\big)$ *is a complete metric space if and only if*
$\liminf\{D(\lambda_k, \overline{0})\} > 0$.

Proof: Let $\{u^i\}$ be a Cauchy sequence of fuzzy numbers in $\Gamma(F, \lambda)$. Therefore for given $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d_\lambda(u^i, u^j) = \sup D_\lambda\big(u_k^{(i)}, u_k^{(j)}\big) < \varepsilon$ for all $i, j \geq n_0$.
Since this is true for all k, $D_\lambda\big(u_k^{(i)}, u_k^{(j)}\big) < \varepsilon$ for all $i, j \geq n_0$.
This implies that

$$\sup_k \{ \sup_{\alpha\in[0,1]} \max\big\{|\lambda_k^-(\alpha)|^{1/k}|u_k^{(i)-}(\alpha) - u_k^{(j)-}(\alpha)|^{1/k}, \\ |\lambda_k^+(\alpha)|^{1/k}|u_k^{(i)+}(\alpha) - u_k^{(j)+}(\alpha)|^{1/k}\} < \varepsilon \tag{5}$$

Let $L = \liminf\{D(\lambda_k, \overline{0})\}$

$$= \liminf \Big\{ \sup_{\alpha\in[0,1]} \max\big\{|\lambda_k^-(\alpha)|^{1/k}, |\lambda_k^+(\alpha)|^{1/k}\big\}\Big\} \tag{6}$$

Using (5) and (6) we get

$$|u_k^{(i)-}(\alpha) - u_k^{(j)-}(\alpha)|^{1/k} < \frac{\varepsilon}{L} \text{ and} \tag{7}$$
$$|u_k^{(i)+}(\alpha) - u_k^{(j)+}(\alpha)|^{1/k} < \frac{\varepsilon}{L} \text{ for all } i, j \geq n_0 \tag{8}$$

Hence $\{u_k^{(i)}\}$ is a Cauchy sequence in E' and since (E', D) is complete,

$$\{u_k^{(i)}\} \to u_k \quad \text{as } i \to \infty \tag{9}$$

Hence $D(u_k^{(i)}, u_k) < \frac{\varepsilon}{L}$ for all $i, j \geq n_0$.
Letting $j \to \infty$ in (7) we get

$$|u_k^{(i)-}(\alpha) - u_k^-(\alpha)|^{1/k} < \frac{\varepsilon}{L} \quad \text{and}$$
$$|u_k^{(i)+}(\alpha) - u_k^+(\alpha)|^{1/k} < \frac{\varepsilon}{L}$$

Now

$$|\lambda_k^-(\alpha)|^{1/k}|\,|u_k^{(i)-}(\alpha) - u_k^-(\alpha)|^{1/k} < \varepsilon \quad \text{and}$$
$$|\lambda_k^+(\alpha)|^{1/k}|\,|u_k^{(i)+}(\alpha) - u_k^+(\alpha)|^{1/k} < \varepsilon$$

Hence

$$\sup_k \sup_{\alpha\in[0,1]} \max\Big\{|\lambda_k^-(\alpha)|^{1/k}|\,|u_k^{(i)-}(\alpha) - u_k^-(\alpha)|^{1/k}, \\ |\lambda_k^+(\alpha)|^{1/k}|\,|u_k^{(i)+}(\alpha) - u_k^+(\alpha)|^{1/k}\Big\} < \varepsilon$$

Thus $u_k^{(i)} \to u$ in $(\Gamma(F,\lambda), d_\lambda)$.
Since each (u^i) is in $\Gamma(F)$ we have

$$D(u_k^{(i)}, \overline{0}) < \frac{\varepsilon}{L}. \tag{10}$$

Using (9) and (10),

$$\begin{aligned} D(\lambda_k u_k, \overline{0}) &\le D(\lambda_k, \overline{0}) D(u_k, \overline{0}) \quad (See\ [11]) \\ &\le D(\lambda_k, \overline{0}) \{D(u_k, \overline{0}) + D(u_k^{(i)}, \overline{0})\} \\ &\le L\left(\frac{\varepsilon}{L}, \frac{\varepsilon}{L}\right) \end{aligned}$$

Hence $u \in \Gamma(F,\lambda)$. Thus $\Gamma(F,\lambda)$ is complete.
Conversely suppose $\liminf\{D(\lambda_k, \overline{0})\} = 0$.
Then there exist a subsequence $\{D(\lambda_k, \overline{0}\}$ which is steadily decreasing and tends to zero.
Consider a sequence $\{P_n\}$ of polynomials where $P_n(x) = 1 + x^{n_1} + x^{n_2} + \cdots + + x^{n_k}$. Clearly this sequence is a Cauchy sequence in $(\Gamma(F,\lambda),\ d_\lambda)$.
But it fails to converge to a point in $(\Gamma(F,\lambda),\ d_\lambda)$.

This completes the proof. ■

We now define the subsequences $\chi(F)$ and $\Gamma(F, 1, d)$ and we show that they are complete.

Definition III.7. *For each fixed k, we define a fuzzy metric*

$$D_\chi(u_k, v_k) = \sup_{\alpha\in[0,1]} \max\left\{|k! u_k^-(\alpha) - k! v_k^-(\alpha)|^{1/k}, |k! u_k^+(\alpha) - k! v_k^+(\alpha)|^{1/k}\right\}$$

where $u = (u_k)$ and $v = (v_k)$ are sequences of fuzzy numbers and we can easily see that (E', D_χ) is complete.

Definition III.8. *The subspace $\chi(F)$ of $\Gamma(F)$ is defined by*

$$\chi(F) = \Big\{u = (u_k) \in \Gamma(F) : \lim_{k\to\infty} D_\chi(u_k, \overline{0}) = 0\Big\}.$$

In other words given $\varepsilon > 0$ there exists a positive integer $n_0 \in \mathbb{N}$ such that $D_\chi(u_k, \overline{0}) < \varepsilon$ for all $k \ge n_0$.

Theorem III.9. *$\chi(F)$ is a complete metric space (See [10]).*

Definition III.10. *For each fixed k we define a fuzzy metric by*

$$\overline{D}(u_k, v_k) = \sup_{\alpha\in[0,1]} \max\left\{k|u_k^-(\alpha) - v_k^-(\alpha)|^{1/k}, k|u_k^+(\alpha) - v_k^+(\alpha)|^{1/k}\right\}$$

where $u = (u_k)$ and $v = (v_k)$ are sequences of fuzzy numbers and it is clear that $(E', \overline{D})$ is complete.

Definition III.11. *We define the subsequence $\Gamma(F, 1, d)$ of $\Gamma(F)$ by*

$$\Gamma(F, 1, d) = \left\{u = (u_k) \in \Gamma(F) : \lim_{k\to\infty} \overline{D}(u_k, \overline{0}) = 0\right\}$$

In other words given $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $\overline{D}(u_k, \overline{0}) < \varepsilon$.

Theorem III.12. *$\Gamma(F, 1, d)$ is a complete metric space with respect to the metric $\overline{d}(u, v) \sup \overline{D}(u_k, v_k)$.*

Theorem III.13. *$\chi(F)$ is a proper closed subspace of $\Gamma(F, 1, d)$.*

Proof: Consider the sequence (u_k) defined by $(u_k) = \left(\frac{1}{k!}\right)$ where $k \in E'$.
Then $(u_k) \in \Gamma(F, 1, d)$ but $(u_k) \notin \chi(F)$.
Therefore $\chi(F)$ is a proper subspace of $\Gamma(F, 1, d)$.
Let $u \in \Gamma(F, 1, d)$ be a limit point of $\chi(F)$.
Then there exist a sequence (u^i) in $\chi(F)$ such that $u^i \to u$.
Therefore for given $\varepsilon > 0$ there exist n_0 such that

$$\overline{d}(u', u) = \sup_k (u_k^{(i)}, u) < \varepsilon \quad \text{for all } k \ge n_0.$$

This implies that $k[|u_k^{(i)-}(\alpha) - u_k^-(\alpha)|]^{1/k} < \varepsilon$ and $k[|u_k^{(i)-}(\alpha) - u_k^+(\alpha)|]^{1/k} < \varepsilon$ for all $k \ge n_0$.
Now our aim is to prove $u \in \chi(F)$.

$$\begin{aligned} [\angle k|u_k^-(\alpha)|]^{1/k} &\le [\angle k|u_k^{(i)}(\alpha)|]^{1/k} \\ &\quad + [\angle k|u_k^{(i)-}(\alpha) - u_k^-(\alpha)|]^{1/k} \\ &\le [\angle k|u_k^{(i)}(\alpha)|]^{1/k} + (\angle k)^{1/k}\frac{\varepsilon}{k} \end{aligned}$$

Therefore $u \in \chi(F)$ and is closed. ■

References

[1] L.A. Zadeh, Fuzzy sets, Inf. Control, **8** (1965) 338–353.

[2] Matloka sequences of fuzzy numbers, BUSEFAL, **28** (1986) 28–37.

[3] M. Mursaleen, M. Bagarir, On some sequence spaces of fuzzy numbers, Indian J.Pure Appl. Math., **34(9)** (2003) 1351–1357.

[4] S. Nanda, On sequences of fuzzy numbers, Fuzzy Sets and Systems, **33** (1989) 123–126.

[5] F. Nuray and E. Savas, Statistical convergence of sequences of fuzzy numbers, Math Slovaca, **45(3)** (1995) 269–273.

[6] B.C Tripathy and Dutta, Fuzzy real valued double sequence spaces, Soochow J. Math., **32(4)** 509–520, 2006.

[7] Ganapathy Iyer, On the space of Integral functions, Journal of Indian Mathematical Society New series, **12** (1948) 13–30.

[8] J. Kavikumar, Azme Bin Khamis and R. Kandasamy, Fuzzy Entire sequence spaces, International Journal of Mathematics and Mathematical Sciences, **2007** article ID 58368, (9 pages).

[9] Subramanian and Metin Basarir, The Orlicz space of Entire sequence of fuzzy numbers, Tamsui Oxford Journal of Mathamatical Sciences, **24(1)** (2008) 109–122.

[10] P. Diamond, P. Kloeden, Metric spaces of fuzzy sets, Fuzzy sets and Systems, **35** (1990) 241–249.

[11] O. Talo, Feyzi Basar, Determination of the cluals of classical sets of sequences of fuzzy numbers and related matrix transformations, Computer and Mathematics with Applications, **58** (2009) 717–733.