

# Some results on parallel alternating methods

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*Abstract*—In this paper, we investigate two parallel alternating methods for solving the system of linear equations  $Ax = b$  and give convergence theorems for the parallel alternating methods when the coefficient matrix is a nonsingular H-matrix. Furthermore, we give one example to show our results.

*Keywords*—nonsingular H-matrix, parallel alternating method, convergence.

## I. INTRODUCTION

**F**OR the large system of linear equations

$$Ax = b, \quad (1)$$

where  $A$  is a nonsingular square matrix of order  $n$ ,  $x, b \in R^n$ . Benzi and Szyld [1] analyzed the following alternating method:

Given an initial vector  $x^{(0)}$ , for  $k = 0, 1, 2, \dots$ ,

$$x^{(k+\frac{1}{2})} = M^{-1}Nx^{(k)} + M^{-1}b,$$

$$x^{(k+1)} = P^{-1}Qx^{(k+\frac{1}{2})} + P^{-1}b,$$

where  $A = M - N = P - Q$  are two splittings of  $A$ . They proved its convergence under certain conditions when the coefficient matrix  $A$  is a monotone matrix or a symmetric positive definite matrix.

In paper [2], Climent and Perea introduced two parallel alternating iterative methods.

Assume that

$$A = M_l - N_l = P_l - Q_l, \quad l = 1, 2, \dots, p, \quad (2)$$

where  $M_l$  and  $P_l$  nonsingular matrices;  $E_l$  satisfy  $\sum_{l=1}^p E_l = I$  ( $I$  is an identity matrix), where  $E_l$  are diagonal and  $E_l \geq 0$ .

Method 1: Let  $x^{(0)}$  be a starting vector,  $\varepsilon > 0$  is a given precision. For  $k = 1, 2, \dots$ ,

$$x_l^{(k+\frac{1}{2})} = (M_l^{-1}N_l)^{\mu(k,l)}x^{(k)} + \sum_{i=0}^{\mu(k,l)-1} (M_l^{-1}N_l)^i M_l^{-1}b,$$

$$x_l^{(k+1)} = (P_l^{-1}Q_l)^{\nu(k,l)}x_l^{(k+\frac{1}{2})} + \sum_{i=0}^{\nu(k,l)-1} (P_l^{-1}Q_l)^i P_l^{-1}b,$$

$$x^{(k+1)} = \sum_{l=1}^p E_l x_l^{(k+1)},$$

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If  $\|x^{(k+1)} - x^{(k)}\| < \varepsilon$ , then quit.

It is easy to notice that the iterative matrix of Method 1 is

$$T = \sum_{l=1}^p E_l (P_l^{-1}Q_l)^{\nu(k,l)} (M_l^{-1}N_l)^{\mu(k,l)}.$$

Method 2: Let  $x^{(0)}$  be a starting vector,  $\varepsilon > 0$  is a given precision. For  $k = 1, 2, \dots$ ,

$$\begin{aligned} x^{(k+\frac{1}{2})} &= \sum_{l=1}^p E_l (M_l^{-1}N_l)^{\mu(k,l)} x^{(k)} \\ &\quad + \sum_{l=1}^p E_l \left[ \sum_{i=0}^{\mu(k,l)-1} (M_l^{-1}N_l)^i M_l^{-1} \right] b, \\ x^{(k+1)} &= \sum_{l=1}^p F_l (P_l^{-1}Q_l)^{\nu(k,l)} x_l^{(k+\frac{1}{2})} \\ &\quad + \sum_{l=1}^p F_l \left[ \sum_{i=0}^{\nu(k,l)-1} (P_l^{-1}Q_l)^i P_l^{-1} \right] b. \end{aligned}$$

If  $\|x^{(k+1)} - x^{(k)}\| < \varepsilon$ , then quit.

It is easy to notice that the iterative matrix of Method 2 is

$$S = \left[ \sum_{l=1}^p F_l (P_l^{-1}Q_l)^{\nu(k,l)} \right] \left[ \sum_{l=1}^p E_l (M_l^{-1}N_l)^{\mu(k,l)} \right].$$

In this paper, we give convergence theorems for the parallel alternating methods when the coefficient matrix is a nonsingular H-matrix.

## II. PRELIMINARIES

Let  $A \in R^{n \times n}$ . We denote by  $A \geq 0$  a nonnegative matrix,  $|A|$  the absolute value of matrix  $A$ , and  $\rho(A)$  the spectral radius of  $A$ .

**Definition 2.1** Let  $A = B - C$  be a splitting of  $A$ . If  $B^{-1} \geq 0, B^{-1}C \geq 0$ , then  $A = B - C$  is a weak regular splitting[3]. If  $B^{-1} \geq 0, C \geq 0$ , then  $A = B - C$  is a regular splitting[4]. If  $B$  is an M-matrix and  $C \geq 0$ , then  $A = B - C$  is an M-splitting[5].

In paper [2], a weak regular splitting is also called a weak nonnegative splitting of the first type.

It's obvious that an M-splitting is a regular splitting and a regular splitting is a weak regular splitting.

**Definition 2.2**([6]) Let  $A \in R^{n \times n}$ .  $A = M - N$  ( $M, N \in R^{n \times n}$ ) is called as an H-splitting if  $\langle M \rangle - |N|$  is an M-matrix. If  $\langle A \rangle = \langle M \rangle - |N|$ , then  $A = M - N$  is called as an H-compatible splitting.

## III. CONVERGENCE THEOREMS

In this section, we give convergence theorems for the parallel alternating methods when the coefficient matrix is a nonsingular H-matrix.

**Lemma 3.1**[2] Let  $A \in R^{n \times n}$  and  $A^{-1} \geq 0$ . If  $A = M_l - N_l = P_l - Q_l$  ( $l = 1, 2, \dots, p$ ) are all weak nonnegative splittings of the first type, then

$$\rho(T) < 1,$$

where

$$T = \sum_{l=1}^p E_l(P_l^{-1}Q_l)^{\nu(k,l)}(M_l^{-1}N_l)^{\mu(k,l)}.$$

**Lemma 3.2**[2] Let  $A \in R^{n \times n}$  and  $A^{-1} \geq 0$ . If  $A = M_l - N_l = P_l - Q_l$  ( $l = 1, 2, \dots, p$ ) are all weak nonnegative splittings of the first type, then

$$\rho(S) < 1,$$

where

$$S = \left[ \sum_{l=1}^p F_l(P_l^{-1}Q_l)^{\nu(k,l)} \right] \left[ \sum_{l=1}^p E_l(M_l^{-1}N_l)^{\mu(k,l)} \right].$$

**Lemma 3.3**[7] If  $A \in R^{n \times n}$  is a nonsingular H-matrix, then  $|A^{-1}| \leq \langle A \rangle^{-1}$ .

**Theorem 3.1** Let  $A \in R^{n \times n}$  be a nonsingular H-matrix,

$$A = M_l - N_l = P_l - Q_l \quad (l = 1, 2, \dots, p)$$

are H-splittings while  $B \in R^{n \times n}$  be a nonsingular M-matrix,

$$B = \langle M_l \rangle - |N_l| = \langle P_l \rangle - |Q_l| \quad (l = 1, 2, \dots, p),$$

then Method 1 converges to the unique solution of (??) for any starting vector  $x^{(0)}$ .

**Proof:** We will show that  $\rho(T) < 1$ .

It is obvious that  $\rho(T) < 1$  if  $\rho(|T|) < 1$ . From

$$\langle M_l \rangle - |N_l| = \langle P_l \rangle - |Q_l| = B$$

is a nonsingular M-matrix. So  $\langle M_l \rangle, \langle P_l \rangle$  ( $l = 1, 2, \dots, p$ ) are all M-matrices, and

$$\langle M_l \rangle - |N_l| = \langle P_l \rangle - |Q_l| = B \quad (l = 1, 2, \dots, p)$$

are M-splittings of  $B$ . So  $M_l, P_l$  ( $l = 1, 2, \dots, p$ ) are all H-matrices. Moreover,

$$|M_l^{-1}| \leq \langle M_l \rangle^{-1}, |P_l^{-1}| \leq \langle P_l \rangle^{-1}.$$

Thus we obtain

$$\begin{aligned} |T| &= \left| \sum_{l=1}^p E_l(P_l^{-1}Q_l)^{\nu(k,l)}(M_l^{-1}N_l)^{\mu(k,l)} \right| \\ &\leq \sum_{l=1}^p E_l(\langle P_l \rangle^{-1} |Q_l|)^{\nu(k,l)} (\langle M_l \rangle^{-1} |N_l|)^{\mu(k,l)} \\ &= \bar{T}. \end{aligned}$$

We use Lemma 3.1 to see immediately that  $\rho(\bar{T}) < 1$ . Therefore,  $\rho(|T|) < 1$  and  $\rho(T) < 1$ , we obtain the conclusion of this theorem.

If  $B = \langle A \rangle$  in Theorem 3.1, then we can obtain the following corollary.

**Corollary 3.1** Let  $A \in R^{n \times n}$  be a nonsingular H-matrix,

$$A = M_l - N_l = P_l - Q_l \quad (l = 1, 2, \dots, p)$$

are H-compatible splittings, then Method 1 converges to the unique solution of (??) for any starting vector  $x^{(0)}$ .

**Theorem 3.2** Let  $A \in R^{n \times n}$  be a nonsingular H-matrix,

$$A = M_l - N_l = P_l - Q_l \quad (l = 1, 2, \dots, p)$$

are H-splittings while  $B \in R^{n \times n}$  be a nonsingular M-matrix,

$$B = \langle M_l \rangle - |N_l| = \langle P_l \rangle - |Q_l| \quad (l = 1, 2, \dots, p),$$

then Method 2 converges to the unique solution of (??) for any starting vector  $x^{(0)}$ .

**Proof:** We will show that  $\rho(S) < 1$ .

It is obvious that  $\rho(S) < 1$  if  $\rho(|S|) < 1$ . From

$$\langle M_l \rangle - |N_l| = \langle P_l \rangle - |Q_l| = B$$

is a nonsingular M-matrix. So  $\langle M_l \rangle, \langle P_l \rangle$  ( $l = 1, 2, \dots, p$ ) are all M-matrices, and

$$\langle M_l \rangle - |N_l| = \langle P_l \rangle - |Q_l| = B \quad (l = 1, 2, \dots, p)$$

are M-splittings of  $B$ . So  $M_l, P_l$  ( $l = 1, 2, \dots, p$ ) are all H-matrices. Moreover,

$$|M_l^{-1}| \leq \langle M_l \rangle^{-1}, |P_l^{-1}| \leq \langle P_l \rangle^{-1}.$$

Thus we obtain

$$\begin{aligned} |S| &= \left| \left[ \sum_{l=1}^p F_l(P_l^{-1}Q_l)^{\nu(k,l)} \right] \left[ \sum_{l=1}^p E_l(M_l^{-1}N_l)^{\mu(k,l)} \right] \right| \\ &\leq \sum_{l=1}^p F_l(\langle P_l \rangle^{-1} |Q_l|)^{\nu(k,l)} \sum_{l=1}^p E_l(\langle M_l \rangle^{-1} |N_l|)^{\mu(k,l)} \\ &= \bar{S}. \end{aligned}$$

We use Lemma 3.2 to see immediately that  $\rho(\bar{S}) < 1$ . Therefore,  $\rho(|S|) < 1$  and  $\rho(S) < 1$ , we obtain the conclusion of this theorem.

If  $B = \langle A \rangle$  in Theorem 3.2, then we can obtain the following corollary.

**Corollary 3.2** Let  $A \in R^{n \times n}$  be a nonsingular H-matrix,

$$A = M_l - N_l = P_l - Q_l \quad (l = 1, 2, \dots, p)$$

are H-compatible splittings, then Method 2 converges to the unique solution of (??) for any starting vector  $x^{(0)}$ .

**Example**

$$A = \begin{bmatrix} 10 & 3 & 8 \\ 4 & 10 & 0 \\ 7 & 3 & 12 \end{bmatrix}$$

is a nonsingular H-matrix. Let

$$M_1 = \begin{bmatrix} 10 & 3 & 5 \\ 5 & 10 & 0 \\ 6 & 5 & 12 \end{bmatrix} \quad M_2 = \begin{bmatrix} 10 & 3 & 6 \\ 5 & 10 & 0 \\ 6 & 5 & 12 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 10 & 3 & 4 \\ 5 & 10 & 0 \\ 6 & 5 & 12 \end{bmatrix} \quad P_2 = \begin{bmatrix} 10 & 3 & 4 \\ 5 & 10 & 0 \\ 5 & 5 & 12 \end{bmatrix}$$

$$N_l = M_l - A, \quad Q_l = P_l - A \quad (l = 1, 2), \quad \mu(k, l) = \nu(k, l) = 2.$$

It's easy to test that

$$\begin{aligned} \langle M_1 \rangle - |N_1| &= \langle M_2 \rangle - |N_2| = \langle P_1 \rangle - |Q_1| \\ &= \langle P_2 \rangle - |Q_2| = \begin{bmatrix} 10 & -3 & -8 \\ -6 & 10 & 0 \\ -7 & -7 & 12 \end{bmatrix} \end{aligned}$$

is a nonsingular M-matrix, but

$$\begin{bmatrix} 10 & -3 & -8 \\ -6 & 10 & 0 \\ -7 & -7 & 12 \end{bmatrix} \neq \langle A \rangle,$$

so

$$A = M_l - N_l = P_l - Q_l \quad (l = 1, 2)$$

are H-splittings.

Case 1: We choose

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then

$$T = \begin{bmatrix} \frac{48}{7033} & -\frac{89}{5135} & -\frac{237}{10805} \\ -\frac{31}{12518} & \frac{123}{21529} & \frac{40}{9861} \\ -\frac{52}{17127} & \frac{212}{15969} & \frac{150}{9149} \end{bmatrix}$$

$$\rho(T) = \frac{103}{3963} < 1.$$

Case 2: We choose  $E_1 = I/3$ ,  $E_2 = 2I/3$ ,  $F_1 = 3I/4$ ,  $F_2 = I/4$ ,  $l = 1, 2$ , then

$$S = \begin{bmatrix} \frac{191}{31989} & -\frac{119}{7022} & -\frac{231}{12751} \\ -\frac{64}{28429} & \frac{19}{3855} & \frac{25}{6782} \\ -\frac{30}{11587} & \frac{75}{6682} & \frac{133}{8366} \end{bmatrix}$$

$$\rho(S) = \frac{163}{6917} < 1.$$

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