Anti-Homomorphism in Fuzzy Ideals

K. Chandrasekhara Rao and V. Swaminathan

Abstract—The anti-homomorphic image of fuzzy ideals, fuzzy ideals of near-rings and anti ideals are discussed in this note. A necessary and sufficient condition has been established for near-ring anti ideal to be characteristic.

Keywords—Fuzzy Ideals, Anti fuzzy subgroup, Anti fuzzy ideals, Anti homomorphism, Lower α level cut.

I. Introduction

IN 1971, Rosenfeld [11] constituted the elementary concepts of fuzzy subgroupoid, fuzzy ideals and fuzzy subgroups. Biswas [3] introduced the notion of anti fuzzy subgroups. Fuzzy subnear-rings are introduced by Abou-Zaid [1]. He studied fuzzy left (resp. right) ideals of a near-ring and gave some properties of fuzzy prime ideals of a near-ring. In [7], it has been established that homomorphic image of a fuzzy left (resp. right) ideal which has "sup property" is a fuzzy left (resp. right) ideal. In the year 1998, Sung M.H. et al. [12] proved the same result using the level fuzzy subsets and obtained some properties based on near-ring homomorphism. Properties of anti-homomorphic images of near-rings are discussed in [5]. Homomorphic images and pre images of anti fuzzy ideals are investigated by K.H. Kim et al. [9]. The notion of anti homomorphic image and pre image of fuzzy and anti fuzzy ideals are investigated in this paper. Also, near-ring anti homomorphic image and pre image of ideals are obtained.

A. Preliminaries

In this section, review of fuzzy set theoretic concepts are given briefly (for details one can refer [4], [11] and [10]). A fuzzy set μ of a set N is a function $\mu : N \to [0,1]$.

μ will be called a fuzzy left ideal [11], if $\mu(xy) \geq \mu(y)$; a fuzzy right ideal, if $\mu(xy) \geq \mu(x)$; anti fuzzy left ideal [3] if $\mu(xy) \leq \mu(y)$; anti fuzzy right ideal, if $\mu(xy) \leq \mu(x)$;.

Let $f : N \to N'$ be a function and let μ and ν be fuzzy sets in N and N' respectively. Then $f(\mu)$ [11], the image of μ under f is a fuzzy set in N' defined by

$$f(\mu)(y) = \begin{cases} \sup\left(\mu(x) : x \in f^{-1}(y)\right) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{Otherwise} \end{cases}$$

for all $y \in N'$. $f^{-1}(\nu)$ [11], the preimage of ν under f is a fuzzy set in N given by

$$f^{-1}(\nu)(x) = \nu\left(f(x)\right)$$

for all $x \in N$.

Similar to an α level cut [4], we have lower level cut [9] as follows:

Let μ be a fuzzy set in a set N. For $\alpha \in [0,1]$, the lower α level cut of μ is denoted by ${}_{\alpha}N_{\mu}$ and is given by

$${}_{\alpha}N_{\mu} = \{n \in N : \mu(n) \leq \alpha\}.$$

Definition 1.1: [1], [7] Let N be a left near-ring and μ be a non empty fuzzy sub set of N. μ is said to be a fuzzy left N-ideal if

I-1. $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,
I-2. $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in N$,
I-3. $\mu(y+x-y) \geq \mu(x)$ and
I-4. $\mu(xy) \geq \mu(y)$ where $x, y \in N$

are satisfied. If axioms (I-1), (I-2), (I-3) with

I-5. $\mu((x+z)y - xy) \geq \mu(z)$

holds, μ is a fuzzy right N-ideal.

From the definition of right near-ring [6], ideals can be defined as follows:

Definition 1.2: Let N be a right near-ring and μ be a non empty fuzzy sub set of N. μ is said to be a fuzzy left N-ideal if (I-1), (I-2), (I-3) and

I-6. $\mu(xy) \geq \mu(y)$ where $x, y \in N$

are satisfied. If (I-7) is postulated instead of (I-5) of fuzzy N-ideal of left near-ring, μ is a fuzzy right N-ideal where

I-7. $\mu(y(x+z) - yx) \geq \mu(z)$.

Definition 1.3: [9] Let N be a left near-ring and μ be a non empty fuzzy sub set of N. μ is said to be an anti fuzzy left N-ideal if

AI-1. $\mu(x-y) \leq max(\mu(x), \mu(y))$,
AI-2. $\mu(xy) \leq max(\mu(x), \mu(y))$,
AI-3. $\mu(y+x-y) \leq \mu(x)$ and
AI-4. $\mu(xy) \leq \mu(x)$ where $x, y \in N$.

If axioms (AI-1), (AI-2), (AI-3) with the following (AI-5) are satisfied then μ is an anti fuzzy right N-ideal;

AI-5. $\mu((x+z)y - xy) \leq \mu(z)$.

Definition 1.4: Let N be a right near-ring and μ be a non empty fuzzy sub set of N. μ is said to be an anti fuzzy left N-ideal when AI-1 to AI-3 along with

AI-6. $\mu(xy) \leq \mu(y)$ where $x, y \in N$

are postulated. If axioms (AI-1), (AI-2), (AI-3) with (AI-7) are satisfied then μ is an anti fuzzy right N-ideal;

AI-7. $\mu(y(x+z) - yx) \leq \mu(z)$.

Recall that, a function $f : N \to N'$ of near-rings is called an ***anti-homomorphism*** [5] when

1. $f(n+m) = f(m) + f(n)$
2. $f(nm) = f(m)f(n)$, for all $n, m \in N$.

A surjective anti-homomorphism is called an **anti-epimorphism** ($\cong$).

II. Main Results

A. Fuzzy Ideals

It is to be noted that the anti homomorphic image pre-image of a fuzzy groupoid is again a fuzzy groupoid. Where as,

Result 2.1: An anti homomorphic pre-image of a right (left) ideal is a left (right) ideal.

Proof: Let ν be a fuzzy left ideal. Then

$$\begin{aligned} \mu(xy) &= \nu(f(xy)) \\ &= \nu(f(y)f(x)) \\ &\geq \nu(f(x)) \\ &= \mu(x), \end{aligned}$$

a right ideal. When ν is a fuzzy right ideal,

$$\begin{aligned} \mu(xy) &= \nu(f(xy)) \\ &= \nu(f(y)f(x)) \\ &\geq \nu(f(y)) \\ &= \mu(y), \end{aligned}$$

a left ideal. □

Result 2.2: Anti-homomorphic image of a fuzzy left (right) ideal, with spermium property, is a fuzzy right (left) ideal.

Proof: Let μ be a fuzzy left ideal with sup property. Given $f(x), f(y)$ in $f(N)$, let $x_0 \in f^{-1}[f(x)], y_0 \in f^{-1}[f(y)]$ be such that

$$\mu(x_0) = \sup_{t \in f^{-1}[f(x)]} \mu(t), \quad \mu(y_0) = \sup_{t \in f^{-1}[f(y)]} \mu(t)$$

respectively. Then

$$\begin{aligned} \nu[f(x)f(y)] &= \nu[f(yx)] \\ &= \sup_{t \in f^{-1}[f(yx)]} \mu(t) \\ &\geq \mu(y_0 x_0) \\ &\geq \mu(x_0) \\ &= \sup_{t \in f^{-1}[f(x)]} \mu(t) \\ &= \nu[f(x)], \end{aligned}$$

implies ν is a fuzzy right ideal. Similarly, when μ is a fuzzy right ideal with above property, we have ν is a fuzzy left ideal. □

In sequel to the above results, the following also can be established for the case of anti fuzzy left (right) ideals.

Result 2.3: An anti homomorphic pre-image of an anti fuzzy right (left) ideal is an anti fuzzy left (right) ideal.

Result 2.4: An anti homomorphic image of an anti fuzzy right (left) ideal with sup property, is an anti fuzzy left (right) ideal.

B. Fuzzy Ideals in Near-rings

Result 2.5: ([5] Theorem 2.2) Anti homomorphic image of a right near-ring (left near-ring) is a left near-ring (right near-ring).

Result 2.6: Let $f : N \to N'$ be an anti-epimorphism of near-rings. If ν is a fuzzy (left/right) ideal in the right (left) near-ring N', then μ, which is $f^{-1}(\nu)$ is a fuzzy (left/right) ideal in the left (right) near-ring N.

Proof:
Let ν be a fuzzy left ideal of right near-ring N'. The proof of conditions (I-1), (I-2) and (I-3) of definition 1.1 are similar to that of proof of [7] Theorem 2.12. For any $x, y \in N$, we have

$$\begin{aligned} \mu(xy) &= \nu(f(xy)) \\ &= \nu(f(y)f(x)) \\ &\geq v(f(y)) \\ &= \mu(y). \end{aligned}$$

Thus μ is a fuzzy left ideal of the left near-ring N. When ν is a right ideal of right near-ring N', for any $x, y, z \in N$ we have,

$$\begin{aligned} \mu((x+z)y - xy) &= \nu(f((x+z)y - xy)) \\ &= \nu(f(y)f(x+z) - f(xy)) \\ &= \nu(f(y)(f(x)+f(z)) - f(y)f(x)) \\ &\geq \nu(f(z)) \\ &= \mu(z), \end{aligned}$$

μ is a fuzzy right ideal of left near-ring N.

Let ν be a right ideal of left near-ring N'. For any $x, y, z \in N$, we have

$$\begin{aligned} \mu(y(x+z) - yx) &= \nu(f((x+z)y - xy)) \\ &= \nu(f(x+z)f(y) - f(yx)) \\ &= \nu((f(x)+f(z))f(y) - f(x)f(y)) \\ &\geq \nu(f(z)) \\ &= \mu(z). \end{aligned}$$

Thus μ is a fuzzy left ideal of right near-ring N. Let ν is a fuzzy left ideal of left near-ring N'. Then

$$\begin{aligned} \mu(xy) &= \nu(f(xy)) \\ &= \nu(f(y)f(x)) \\ &\geq \nu(f(x)) \\ &= \mu(x) \end{aligned}$$

for all $x, y \in N$, implies μ is a left ideal of right near ring N. □

Result 2.7: Let $f : N \to N'$ be an anti-epimorphism of near-rings. If μ is a fuzzy (left/right) ideal in the left (right) near-ring N with sup property, then $\nu = f(\mu)$ is a fuzzy (left/right) ideal in the right (left) near-ring N'.

Proof: Let μ be a fuzzy left ideal of the left near-ring N with sup property and ν be the image of μ under f. Let $x_0 \in f^{-1}[f(x)]$, $y_0 \in f^{-1}[f(y)]$ such that

$$\mu(x_0) = \sup_{t\in f^{-1}[f(x)]} \mu(t), \quad \mu(y_0) = \sup_{t\in f^{-1}[f(y)]} \mu(t).$$

$$\begin{aligned} \nu[f(x)f(y)] &= \sup_{t\in f^{-1}[f(yx)]} \mu(t) \\ &\geq \mu(y_0 x_0) \\ &\geq \mu(x_0) \\ &= \sup_{t\in f^{-1}[f(x)]} \mu(t) \\ &= \nu[f(x)]. \end{aligned}$$

That is ν is a fuzzy left ideal of right near-ring N'.

If μ is a fuzzy right ideal of left near-ring. For any $f(z) \in f(N)$, let $z_0 \in f^{-1}[f(z)]$ such that

$$\mu(z_0) = \sup_{t\in f^{-1}[f(z)]} \mu(z).$$

Now,

$$\begin{aligned} & \nu[f\{(x+z)y-(xy)\}] \\ =\ & \nu[f(y)f(x+z)-f(xy)] \\ =\ & \nu[f(y)\{f(x)+f(z)\}-f(y)f(x)] \\ =\ & \sup_{t\in f^{-1}\{f[f(y)\{f(x)+f(z)\}-f(y)f(x)]\}} \mu(t) \\ \geq\ & \mu[y_0\{x_0+z_0\}-y_0x_0] \\ \geq\ & \mu[z_0] \\ =\ & \sup_{t\in f^{-1}[f(z)]} \mu(t) \\ =\ & \nu[f(z)]. \end{aligned}$$

That is, ν is a fuzzy right ideal of right near-ring.

The image and pre-image of the fuzzy ideal of a fuzzy right near-ring N can be proved to be the fuzzy ideal of a left near-ring N'. □

C. Anti Fuzzy Ideals in Near-rings

Definition 2.8: A left N-ideal A of a near-ring is said to be characteristic [2], [8] if

$$f(A) = A \ \ \forall f \in Aut(N)$$

where $Aut(N)$ is set of all automorphism of N.

Anti fuzzy left N-ideal of μ of a near-ring N is said to be anti fuzzy characteristic if

$$\mu f_{(x)} = \mu(x) \quad \forall x \in N, f \in Aut(N).$$

Lemma 2.9: Let μ be an anti fuzzy left N-ideal of a near-ring N and let $x \in N$. Then $\mu(x) = s$ if and only if $x \in {}_sN_\mu$ and $x \notin {}_tN_\mu$ $\forall s > t$.

Proof is obvious.

The proof of the following theorem is analogous to the proof of theorem 3.9 [2], [8].

Theorem 2.10: Let μ be an anti fuzzy N-ideal of a near-ring N. Then each lower α level left N-ideal of μ is characteristic iff μ is an anti fuzzy characteristic of N.

K. H. Kim et al. [9] proved the following theorems:

Result 2.11 ([9], Theorem 3.19 (1)): Let $f : N \to N'$ be an epimorphism of near-rings. Let ν be an anti-fuzzy left N'-ideal and μ be the pre-image of ν under f. Then μ is an anti-fuzzy left N-ideal.

Result 2.12 ([9], Theorem 3.19 (2)): Let $f : N \to N'$ be a surjective homomorphism of near-rings. If μ is an anti fuzzy left N'-ideal then $f^{-1}(\mu)$ is an anti fuzzy left N-ideal.

Result 2.13: Let $f : N \to N'$ be an anti-epimorphism of near-rings. Let ν be an anti-fuzzy (left/right) ideal of right (left) near-ring N' then μ, the pre-image of ν under f, is an anti-fuzzy (left/right) ideal of left (right) near-ring N.

The proof of AI-4 and AI-5 are similar to the proof of result 2.6.

Result 2.14: Let $f : N \to N'$ be an anti epimorphism. If μ is an anti fuzzy (left/right) ideal of left (right) near-ring N with sup property, $f(\mu)$ is an anti fuzzy (left/right) ideal of right (left) near-ring N'.

The proof of AI-6 and AI-7 are similar to that of result 2.7.

References

[1] S. Abou-Zaid, *On fuzzy subnear-rings and ideals,* Fuzzy Sets and Systems 44, 139 - 146, 1991.
[2] M. Akram and K. H. Dar, *On Anti Fuzzy Left h-ideals in Hemirings*, International Mathematical Forum, 2, no. 46, 2295 - 2304, 2007.
[3] R. Biswas, *Fuzzy subgroups and anti fuzzy subgroups,* Fuzzy Sets Systems 44, 121 - 124, 1990.
[4] P. S. Das, *Fuzzzy groups and level subgroups*, J. Math. Anal. and Appl. 84, 264 - 269, 1981.
[5] K. Chandrasekhara Rao and V. Swaminathan, *Anti-homomorphisms in Near-Rings* Jour. of Inst. of Maths. Comp. Sciences (Maths. Ser.) Vol. 21, No. 2 , 83 - 88, 2008.
[6] Günter Pilz, *Near-rings*, The Theory and applications, North-Holland Publishing company, Amsterdam, New York, Oxford, 1983.
[7] S. D. Kim and H. S. Kim, *On fuzzy ideals of near-rings,* Bull. Korean Math. Soc. 33, 593 - 601, 1996.
[8] K. H . Kim and Y. B. Jun, *A note on fuzzy R-subgroups of near-rings*, Soochow Journal of Mathematics, Vol. 28, no. 4, 330 - 346, 2002.
[9] K. H . Kim and Y. B. Jun, *On Anti Fuzzy Ideals in Near-Rings,* Iranian Jr. of Fuzzy Systems, Vol. 2, No. 2, 71 - 80, 2005.
[10] Mohamed Asaad, *Groups and fuzzy subgroups*, Fuzzy Sets and Systems 39, 323 - 328, 1991.
[11] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. 35, 512 - 517, 1971.
[12] Sung Min Hong, Young Bae Jun and Hee Sik Kim, *Fuzzy Ideals in Near-rings*, Bull. Korean Math. Soc. 35, pp 455 - 464, 1998.
[13] H. Kopka and P. W. Daly, *A Guide to LaTeX*, 3rd ed. Harlow, England: Addison-Wesley, 1999.

KC Rao is Professor in Mathematics, with more than 4 decades of experience, 4 books published by Alpha Science, Oxford and authored more than 125 papers in various national and international journals. He also produced about 25 Ph. D's in Functional Analysis, Topology and Algebra. He is the reviewer of "Zantralblatt MATH" and "Mathematical Reviews".

V. Swaminathan, working as Asst. Professor in the Department of Mathematics, Srinivasa Ramanujan Centre, SASTRA University, Kumbakonam, India. He has published about 7 papers in national and international journals.