

Induced Acyclic Graphoidal Covers in a Graph

K. Ratan Singh, P. K. Das

Abstract—An induced acyclic graphoidal cover of a graph G is a collection ψ of open paths in G such that every path in ψ has atleast two vertices, every vertex of G is an internal vertex of at most one path in ψ , every edge of G is in exactly one path in ψ and every member of ψ is an induced path. The minimum cardinality of an induced acyclic graphoidal cover of G is called the induced acyclic graphoidal covering number of G and is denoted by $\eta_{ia}(G)$ or η_{ia} . Here we find induced acyclic graphoidal cover for some classes of graphs.

Keywords—Graphoidal cover, Induced acyclic graphoidal cover, Induced acyclic graphoidal covering number.

I. INTRODUCTION

A graph is a pair $G = (V, E)$, where V is the set of vertices and E is the set of edges. Here, we consider only nontrivial, simple, finite and connected graphs. The order and size of G are denoted by p and q respectively. The concept of graphoidal cover was introduced by B.D. Acharya and E. Sampathkumar [1] and the concept of induced acyclic graphoidal cover was introduced by S. Arumugam [4]. The reader may refer [3], [5] and [6] for the terms not defined here.

Definition I.1. [1] A graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions:

- (i) Every path in ψ has atleast two vertices.
- (ii) Every vertex of G is an internal vertex of at most one path in ψ .
- (iii) Every edge of G is in exactly one path in ψ .

The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by $\eta(G)$.

Definition I.2. [4] An induced graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions:

- (i) Every path in ψ has atleast two vertices.
- (ii) Every vertex of G is an internal vertex of at most one path in ψ .
- (iii) Every edge of G is in exactly one path in ψ .
- (iv) Every member of ψ is an induced cycle or an induced path.

The minimum cardinality of an induced graphoidal cover of G is called the induced graphoidal covering number of G and is denoted by $\eta_i(G)$ or η_i .

Let ψ be a graphoidal cover of G and Z be a cycle of G . Then for any edge e of Z , the family $\psi_1 = (\psi \setminus \{Z\}) \cup$

$\{Z - e\} \cup \{e\}$ is again a graphoidal cover of G . Thus one can successively break up every cycle member of ψ into paths eventually yielding a graphoidal cover ψ_1 of G that has only path members. Motivated by this observation, S. Arumugam and Suresh Suseela [2] introduced the concept of acyclic graphoidal cover and acyclic graphoidal covering number of a graph.

Definition I.3. [2] A graphoidal cover ψ of a graph G is called an acyclic graphoidal cover if every member of ψ is a path. The minimum cardinality of an acyclic graphoidal cover of G is called the acyclic graphoidal covering number of G and is denoted by $\eta_a(G)$ or η_a .

Definition I.4. [4] A graphoidal cover ψ of a graph G is called an induced acyclic graphoidal cover if every member of ψ is an induced path. The minimum cardinality of an induced acyclic graphoidal cover of G is called the induced acyclic graphoidal covering number of G and is denoted by $\eta_{ia}(G)$ or η_{ia} .

Definition I.5. Let ψ be a collection of internally edge disjoint paths in G . A vertex of G is said to be an internal vertex of ψ if it is an internal vertex of some path in ψ , otherwise it is called an external vertex of ψ .

II. MAIN RESULTS

The following result for graphoidal covering number also holds for induced acyclic graphoidal covering number.

Theorem II.1. [3] For any induced acyclic graphoidal cover ψ of a (p, q) - graph G , let t_ψ denote the number of external vertices of ψ and let $t = \min t_\psi$, where the minimum is taken over all induced acyclic graphoidal covers ψ of G then $\eta_{ia}(G) = q - p + t$.

Corollary II.2. For any graph G , $\eta_{ia}(G) \geq q - p$. Moreover, the following are equivalent

- (i) $\eta_{ia}(G) = q - p$.
- (ii) There exists an induced acyclic graphoidal cover of G without external vertices.
- (iii) There exists a set Q of internally disjoint and edge disjoint induced acyclic graphoidal path without exterior vertices (From such a set Q of paths, the required induced acyclic graphoidal cover can be obtained by adding the edges which are not covered by the paths in Q).

Corollary II.3. If there exists an induced acyclic graphoidal cover ψ of a graph G such that every vertex of G with degree atleast two is internal to ψ , then ψ is a minimum induced acyclic graphoidal cover of G and $\eta_{ia}(G) = q - p + n$, where n is the number of pendant vertices of G .

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Corollary II.4. Since every graphoidal cover of a tree T is also an induced acyclic graphoidal cover of T , we have $\eta_{ia}(T) = n - 1$, where n is the number of pendant vertices of T .

Theorem II.5. Let G be a complete graph K_p . Then $\eta_{ia}(K_p) = q$.

Proof: The result follows from the fact that every member in an induced acyclic graphoidal cover ψ of K_p is an edge. ■

Theorem II.6. Let G be a complete bipartite graph $K_{m,n}$, then

(i) $\eta_{ia}(K_{1,n}) = n - 1, \quad n \geq 2.$

(ii) $\eta_{ia}(K_{2,n}) = q - p + 2, \quad n \geq 2.$

(iii) $\eta_{ia}(K_{3,n}) = \begin{cases} q - p + 2 & \text{if } n = 3, 4, 5; \\ q - p & \text{if } n \geq 6. \end{cases}$

(iv) $\eta_{ia}(K_{m,n}) = q - p$ if $m, n \geq 4.$

Proof: Let $X = \{v_1, v_2, v_3, \dots, v_m\}$ and $Y = \{w_1, w_2, w_3, \dots, w_n\}$ be a bipartition of $K_{m,n}$.

(i). Since $K_{1,n}$ is a tree with n pendant vertices. Hence $\eta_{ia}(K_{1,n}) = n - 1$.

(ii). When $n \geq 2$. Let $X = \{v_1, v_2\}$ and $Y = \{w_1, w_2, w_3, \dots, w_n\}$ be a bipartition of $K_{2,n}$.

Let $P_i = (v_1, w_i, v_2), i = 1, 2, \dots, n$. Then $\psi = \{P_i | i = 1, 2, \dots, n\}$ is an induced acyclic graphoidal cover of $K_{2,n}$ and $|\psi| = q - p + 2$. Hence $\eta_{ia}(K_{2,n}) \leq q - p + 2$. Further, for any induced acyclic graphoidal cover ψ of $K_{2,n}$ atleast two vertices are external vertices so that $t \geq 2$. Hence $\eta_{ia}(K_{2,n}) = q - p + 2$.

(iii). When $n = 3, 4, 5$. Let $X = \{v_1, v_2, v_3\}$ and $Y = \{w_1, w_2, w_3, \dots, w_n\}$ be a bipartition of $K_{3,n}$.

Let $P_i = (v_1, w_i, v_2), i = 1, 2, \dots, 5$ and $Q = (w_1, v_3, w_2)$. Then $\psi = \{P_i, Q\} \cup S, i = 1, 2, \dots, 5$, where S is the set of edges not covered by the paths $P_i, i = 1, 2, 3, 4, 5$ and Q is an induced acyclic graphoidal cover of $K_{3,n}$ and $|\psi| = q - p + 2$. Hence $\eta_{ia}(K_{3,n}) \leq q - p + 2$. Further, for any induced acyclic graphoidal cover ψ of $K_{3,n}$ atleast two vertices are external vertices so that $t \geq 2$. Hence $\eta_{ia}(K_{3,n}) = q - p + 2$.

When $n \geq 6$. Let $X = \{v_1, v_2, v_3\}$ and $Y = \{w_1, w_2, w_3, \dots, w_n\}$ be a bipartition of $K_{3,n}$.

Let $P_1 = (w_1, v_1, w_2), P_2 = (w_3, v_2, w_4), P_3 = (w_5, v_3, w_6), P_4 = (v_2, w_1, v_3), P_5 = (v_2, w_2, v_3), P_6 = (v_1, w_3, v_3), P_7 = (v_1, w_4, v_3), P_{i+3} = (v_1, w_i, v_2), i = 5, 6, \dots, n$. Then $\psi = \{P_1, P_2, \dots, P_{i+3}\} \cup S$, where S is the set of edges not covered by $P_1, P_2, P_3, \dots, P_{n+3}$ is an induced acyclic graphoidal cover of $K_{3,n}$ and every vertex is an internal vertex of some path in ψ . Hence, $\eta_{ia}(K_{3,n}) = q - p$.

(iv). When $m, n \geq 4$. Let $X = \{v_1, v_2, \dots, v_m\}$ and $Y = \{w_1, w_2, w_3, \dots, w_n\}$ be a bipartition of $K_{m,n}$.

Let $P_1 = (w_1, v_1, w_2), P_2 = (w_2, v_2, w_3), P_3 = (w_3, v_3, w_4), P_4 = (w_1, v_4, w_4), P_5 = (v_2, w_1, v_3), P_6 = (v_3, w_2, v_4), P_7 = (v_1, w_3, v_4), P_8 = (v_1, w_4, v_2), Q_{i-4} = (v_1, w_i, v_2), i = 5, 6, \dots, n$,

$R_{j-4} = (w_1, v_j, w_2), j = 5, 6, \dots, m$. Then $\psi = \{P_l, Q_{i-4}, R_{j-4}\} \cup S, l = 1, 2, \dots, 8, i = 5, 6, \dots, m; j = 5, 6, \dots, m$ and S is the set of edges not covered by P_l, Q_{i-4}, R_{j-4} is an induced acyclic graphoidal cover of $K_{m,n}$ and every vertex is an internal vertex of some path in ψ . Hence, $\eta_{ia}(K_{m,n}) = q - p$. ■

Theorem II.7. For the wheel $W_p = K_1 + C_{p-1}$, we have

$$\eta_{ia}(W_p) = \begin{cases} 6 & \text{if } p = 4; \\ p & \text{if } p \geq 5. \end{cases}$$

Proof: Let $V(W_p) = \{v_0, v_1, \dots, v_{p-1}\}$ and $E(W_p) = \{v_0v_i : 1 \leq i \leq p-1\} \cup \{v_iv_{i+1} : 1 \leq i \leq p-2\} \cup \{v_1v_{p-1}\}$ If $p = 4$ then $W_4 = K_4$ and so $\eta_{ia}(W_p) = 6$.

If $p \geq 5$. Let $P_1 = (v_1, v_2, v_3), P_2 = (v_1, v_0, v_3), P_3 = (v_3, v_4, \dots, v_{p-1}, v_1)$. Then $\psi = \{P_1, P_2, P_3\} \cup S$, where S is the set of edges of W_p not covered by P_1, P_2 and P_3 is an induced acyclic graphoidal cover of W_p and $|\psi| = p$. Hence, $\eta_{ia}(G) \leq p$. On the other hand, for any induced acyclic graphoidal cover ψ of W_p atleast two vertices are external vertices so that $t \geq 2$. Hence $\eta_{ia}(W_p) \geq q - p + 2 = p$. Thus $\eta_{ia}(W_p) = p$. ■

Theorem II.8. If G is a unicyclic graph with n pendant vertices and the unique cycle C_k , and j denote the number of vertices of degree greater than or equal to 3 in C_k then when

(i) $k = 3$

$$\eta_{ia}(G) = \begin{cases} 3 & \text{if } j = 0; \\ n + 2 & \text{if } j = 1; \\ n + 1 & \text{if } j = 2; \\ n & \text{otherwise.} \end{cases}$$

(ii) $k \geq 4$

$$\eta_{ia}(G) = \begin{cases} 2 & \text{if } j = 0; \\ n + 1 & \text{if } j = 1; \text{ or } j = 2 \text{ and the two vertices} \\ & \text{of } \text{deg} \geq 3 \text{ are adjacent in } C_k; \\ n & \text{otherwise.} \end{cases}$$

Proof: Let $C_k = \{v_1, v_2, v_3, \dots, v_k, v_1\}$ be the unique cycle in G .

(i). Case(a). When $j = 0$ then $G = C_3$ so that $\eta_{ia}(G) = 3$.

Case(b). When $j = 1$. Let v_1 be the unique vertex of $\text{deg} \geq 3$ in C_3 . Let $T = G - \{v_1v_2, v_2v_3\}$ be the tree with $n + 1$ pendant vertices so that $\eta_{ia}(T) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of T . Then $\psi = \psi_1 \cup \{v_1v_2, v_2v_3\}$ is an induced acyclic graphoidal cover of G and so $|\psi| = n + 2$. Hence, $\eta_{ia}(G) \leq n + 2$. On the other hand, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast two vertices in C_k are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq q - p + n + 2 = n + 2$.

Case(c). When $j = 2$. Let v_1 and v_2 be the vertices of $\text{deg} \geq 3$ in C_3 . Let $T = G - \{v_2v_3, v_3v_1\}$ be the tree with n pendant vertices so that $\eta_{ia}(T) = n - 1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of T . Then $\psi = \psi_1 \cup \{v_2v_3, v_3v_1\}$ is an induced acyclic graphoidal cover of G and so $|\psi| = n + 1$. Hence, $\eta_{ia}(G) \leq n + 1$. On the other hand, for any induced

acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast one vertex in C_k are external vertices so that $t \geq n + 1$. Hence, $\eta_{ia}(G) = q - p + t \geq q - p + n + 1 = n + 1$.

Case(d). When all the vertices in C_3 are of $\text{deg} \geq 3$. Let $T = G - (v_1 v_2)$ be the tree with n pendant vertices. Let T_1 be the induced subgraph of T formed by v_2 along with vertices connected to v_2 such that v_3 occurs as an pendant vertex. Then T_1 has $n_1 + 1$ pendant vertices so that $\eta_{ia}(T_1) = n_1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of T_1 . Also, $T_2 = T - T_1$ is also a tree with n_2 pendant vertices so that $n_1 + n_2 = n$ and $\eta_{ia}(T_2) = n_2 - 1$. Let ψ_2 be a minimum induced acyclic graphoidal cover of T_2 . Then $\psi = \psi_1 \cup \psi_2 \cup (v_1 v_2)$, is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_{ia}(G) = n$.

(ii). Case(a). When $j = 0$, then $G = C_k$ so that $\eta_{ia}(G) = 2$.

Case(b). When $j = 1$. Let v_1 be the unique vertex of $\text{deg} \geq 3$ in C_k . Let $P = (v_1, v_k, v_{k-1}, \dots, v_4, v_3)$ be an induced path of length atleast 2. Then $T = G - P$ is a tree with $n + 1$ pendant vertices so that $\eta_{ia}(T) = n$, with ψ_1 as a minimum induced acyclic graphoidal cover. Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of G and so $|\psi| = n + 1$. Hence, $\eta_{ia}(G) \leq n + 1$. Further, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast one vertex in C_k are external vertices so that $t \geq n + 1$. Hence, $\eta_{ia}(G) = q - p + t \geq q - p + n + 1 = n + 1$.

When $j = 2$ and the two vertices of $\text{deg} \geq 3$ are adjacent vertices in C_k , the proof is similar to that for $j = 1$.

Case(c). When $j = 2$. Suppose v_1, v_3 are the two non adjacent vertices of $\text{deg} \geq 3$. Let $P = (v_1, v_k, v_{k-1}, \dots, v_4, v_3)$ be an induced path of length atleast 2. Then $T = G - P$ is a tree with n pendant vertices so that $\eta_{ia}(T) = n - 1$, with ψ_1 as a minimum induced acyclic graphoidal cover. Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of G such that every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_{ia}(G) = n$.

When $j \geq 3$. Take two non adjacent vertices v_i and v_j of $\text{deg} \geq 3$ in C_k . let T be the induced subgraph of G containing all vertices on one side of the arc v_i and v_j of G such that these two vertices appear as pendant vertices and T has $n_1 + 2$ pendant vertices so that $\eta_{ia}(T_1) = n_1 + 1$. Let ψ_1 be the minimum induced acyclic graphoidal cover of T . Then $T' = G - T$ is a tree with n_2 pendant vertices so that $n = n_1 + n_2$ and $\eta_{ia}(T') = n_2 - 1$. Let ψ_2 be the minimum induced acyclic graphoidal cover of T' . Then $\psi = \psi_1 \cup \psi_2$, is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_{ia}(G) = n$. ■

Theorem II.9. Let G be a bicyclic graph with n pendant vertices containing a $U(l; m)$ and j be the number of vertices of degree greater than or equal to 3 in $U(l; m)$. Then when (i) $l, m = 3$

$$\eta_{ia}(G) = \begin{cases} 5 & \text{if } G = U(l; m); \\ n + 6 - j & \text{if } 1 \leq j \leq 5. \end{cases}$$

(ii) $l = 3, m \geq 4$

$$\eta_{ia}(G) = \begin{cases} 4 & \text{if } G = U(l; m); \\ n + 4 & \text{if } j = 1; \text{ or } j = 2 \text{ and the other vertex of } \text{deg} \geq 3 \text{ is adjacent to } u_0 \text{ in } C_m; \\ n + 3 & \text{if } j = 2 \text{ and } u_0 \text{ is adjacent to the other vertex of } \text{deg} \geq 3 \text{ in } C_l; \text{ or } j \geq 2 \text{ and all vertices of } \text{deg} \geq 3 \text{ are in } C_m. \\ n + 2 & \text{if } j = 3 \text{ in } C_l; \text{ or } j = 4 \text{ and } C_m \text{ has only one vertex other than } u_0 \text{ of } \text{deg} \geq 3 \text{ which is adjacent to } u_0; \\ n + 1 & \text{otherwise.} \end{cases}$$

(iii) $l, m \geq 4$

$$\eta_{ia}(G) = \begin{cases} 3 & \text{if } G = U(l; m); \\ n + 3 & \text{if } j = 1; \text{ or } j = 2 \text{ and the other vertices of } \text{deg} \geq 3 \text{ is adjacent to } u_0; \text{ or } j = 3 \text{ and the other vertices of } \text{deg} \geq 3 \text{ are adjacent to } u_0 \text{ in } C_m; \\ n + 2 & \text{if } j \geq 2 \text{ and all vertices of } \text{deg} \geq 3 \text{ are in } C_l \text{ or } C_m; \text{ or } j = 2 \text{ and the other vertex of } \text{deg} \geq 3 \text{ is nonadjacent to } u_0; \\ n + 1 & \text{otherwise.} \end{cases}$$

Proof: Let the l -cycle be $C_l = \{u_0, u_1, \dots, u_{l-1}, u_0\}$ and the m -cycle be $C_m = \{u_0, u_l, u_{l+1}, \dots, u_{l+m-2}, u_0\}$ in G .

(i). If $G = U(l; m)$ then $\eta_{ia}(G) = 5$.

Otherwise, Take $G' = G - \{e\}$, where e is an edge with end vertices of degree 2 in G .

If $j = 1$ then G' is a unicyclic graph with $n + 2$ pendent vertices so that $\eta_{ia}(G') = n + 4$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G' . Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 5$. Hence, $\eta_{ia}(G) \leq n + 5$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast four vertices in $U(l; m)$ are external vertices so that $t \geq n + 4$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 4 = n + 5$.

If $j = 2$, similar as above.

If $j = 3$, similar as above if no vertex of C_l except u_0 is of $\text{deg} \geq 3$.

If $j = 3$ each of C_l and C_m has a vertex other than u_0 of $\text{deg} \geq 3$. Let e be an edge in $U(l; m)$ not adjacent to u_0 , then $G' = G - e$ is a unicyclic graph with $n + 1$ pendent vertices so that $\eta_{ia}(G') = n + 2$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G' . Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast two vertices in $U(l; m)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

If $j = 4$, similar as above.

If $j = 5$. Let e be an edge in $U(l; m)$ not adjacent to u_0 . Then $G_1 = G - e$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic

graphoidal cover of G_1 . Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_{ia}(G) = q - p + t = n + 1$.

(ii). Case(a). $G = U(l; m)$. Then $\eta_{ia}(G) = 4$.

Case(b). When $j = 1$. Then $G_1 = G - C_m$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n + 2$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 4$. Hence, $\eta_{ia}(G) \leq n + 4$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast three vertices in $U(l; m)$ are external vertices so that $t \geq n + 3$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 3 = n + 4$.

When $j = 2$ and u_0 is adjacent to the other vertex of $\deg \geq 3$ in C_m . Then $G_1 = G - C_l$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n + 1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_l$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 3 = n + 4$. Hence, $\eta_{ia}(G) \leq n + 4$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast three vertices in $U(l; m)$ are external vertices so that $t \geq n + 3$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 3 = n + 4$.

Case(c). When $j = 2$ and the other vertex of $\deg \geq 3$ is in C_l . Then $G_1 = G - C_m$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n + 1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast two vertices in $U(l; m)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

When $j \geq 2$ and all vertices of $\deg \geq 3$ are in C_m . Then $G_1 = G - C_l$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_l$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 3 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast two vertices in $U(l; m)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Case(d). When $j = 3$ and all vertices of $\deg \geq 3$ are in C_l . Then $G_1 = G - C_m$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 2$. Hence, $\eta_{ia}(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast one vertex in $U(l; m)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

When $j = 4$ and exactly one vertex, say u_l , of $\deg \geq 3$ is adjacent to u_0 in C_m . Let e be an edge not adjacent to u_0 in C_l . Then $G_1 = G - e$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n + 1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta_{ia}(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast one

vertex in $U(l; m)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Case(e). When $j \geq 4$. Let e be an edge in C_l not adjacent to u_0 . Then $G_1 = G - e$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_{ia}(G) = q - p + t = n + 1$.

(iii). Case(a). $G = U(l; m)$. Then $\eta_{ia}(G) = 3$.

Case(b). When $j = 1$. Here, $G_1 = G - C_m$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n + 1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast two vertices in $U(l; m)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Similarly, we can prove for $j = 2$ and the other vertex of $\deg \geq 3$ is adjacent to u_0 .

When $j = 3$. Suppose u_l in C_l and u_{l+m-2} in C_m are of $\deg \geq 3$ and both are adjacent to u_0 . Let P be an induced path $u_l - u_{l+m-2}$ of length atleast two in C_m such that $G_1 = G - P$ is a unicyclic graph with $n + 1$ pendant vertices and so $\eta_{ia}(G_1) = n + 2$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast two vertices in $U(l; m)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Case(c). When $j \geq 2$ and all vertices of $\deg \geq 3$ except u_0 are in C_m or $j = 2$ with the other vertex v of $\deg \geq 3$ is nonadjacent to u_0 in G . Then $G_1 = G - C_l$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_l$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 2$. Hence, $\eta_{ia}(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast one vertex in $U(l; m)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Case(d). When $j = 3$ and $v \in C_l$, $w \in C_m$ of $\deg \geq 3$ are non adjacent to u_0 . Let $P = (u_0, \dots, v)$ be a path of length atleast 2 in C_l such that $G_1 = G - P$ is a unicyclic graph with n pendant vertices and so $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_{ia}(G) = n + 1$.

When $j \geq 4$, Take two non adjacent vertices v_i and v_j of $\deg \geq 3$ in C_l (or C_m). Let T be the induced subgraph of G containing all vertices on one side of the arc $v_i - v_j$ of C_l (or C_m) such that these two vertices appear as pendant vertices and T has $n_1 + 2$ pendant vertices so that $\eta_{ia}(T) = n_1 + 1$. Let ψ_1 be the minimum induced acyclic graphoidal cover of T . Then $G_1 = G - T$ is a unicyclic graph with n_2 pendant

vertices so that $n = n_1 + n_2$ and $\eta_{ia}(G_1) = n_2$. Let ψ_2 be the minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup \psi_2$, is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_{ia}(G) = n + 1$. ■

Theorem II.10. Let G be a bicyclic graph with n pendant vertices containing a $D(l, m; i)$ and j be the number of vertices of degree greater than or equal to 3 in cycles in $D(l, m; i)$. Then when

(i) $l, m = 3$

$$\eta_{ia}(G) = \begin{cases} 5 & \text{if } G = D(l, m; i); \\ n + 7 - j & \text{if } 2 \leq j \leq 6. \end{cases}$$

(ii) $l = 3, m \geq 4$

$$\eta_{ia}(G) = \begin{cases} 4 & \text{if } G = D(l, m; i); \\ n + 4 & \text{if } j = 2; \text{ or } j = 3 \text{ and the third vertex of } \\ & \text{deg} \geq 3 \text{ is adjacent to } u_{l+i-1} \text{ in } C_m; \\ n + 3 & \text{if } j = 3 \text{ and the third vertex of } \text{deg} \geq 3 \text{ is} \\ & \text{in } C_l; \text{ or } j \geq 3 \text{ and all vertices of } \text{deg} \geq 3 \\ & \text{are in } C_m; \text{ or } j = 4 \text{ and } v \text{ in } C_l \text{ and } w \\ & \text{adjacent to } u_{l+i-1} \text{ in } C_m \text{ are of } \text{deg} \geq 3; \\ n + 2 & \text{if } j = 4 \text{ and } C_m \text{ has no vertex of } \text{deg} \geq 3 \\ & \text{other than } u_{l+i-1}; \text{ or } j = 5 \text{ and } C_m \text{ has} \\ & \text{exactly one vertex of } \text{deg} \geq 3 \text{ which is} \\ & \text{adjacent to } u_{l+i-1}; \\ n + 1 & \text{otherwise.} \end{cases}$$

(iii) $l, m \geq 4$

$$\eta_{ia}(G) = \begin{cases} 3 & \text{if } G = D(l, m; i); \\ n + 3 & \text{if } j = 2; \text{ or } j = 3 \text{ and the third vertex of} \\ & \text{deg} \geq 3 \text{ is adjacent to either } u_{l-1} \text{ or} \\ & u_{l+i-1}; \text{ or } j = 4 \text{ and from the vertices of} \\ & \text{deg} \geq 3 \text{ other than } u_{l-1} \text{ and } u_{l+i-1} \text{ one is} \\ & \text{adjacent to } u_{l-1} \text{ in } C_l \text{ and other is} \\ & \text{adjacent to } u_{l+i-1} \text{ in } C_m; \\ n + 2 & \text{if } j \geq 3 \text{ and all vertices of } \text{deg} \geq 3 \text{ are in} \\ & C_l \text{ or } C_m \text{ only; or } j = 3 \text{ and the other} \\ & \text{vertex of } \text{deg} \geq 3 \text{ is adjacent to neither} \\ & u_{l-1} \text{ nor } u_{l+i-1}; \\ n + 1 & \text{otherwise.} \end{cases}$$

Proof: Let $C_l = u_0 u_1 \dots u_{l-1} u_0$, $P_i = u_{l-1} u_l \dots u_{l+i-1}$ and $C_m = u_{l+i-1} u_{l+i} \dots u_{l+m+i-2} u_{l+i-1}$ in G .

(i). If $G = D(l, m; i)$ then $\eta_{ia}(G) = 5$.

Otherwise, Take $G' = G - \{e\}$, where e is an edge with end vertices of degree 2 in G .

If $j = 2$ then G' is a unicyclic graph with $n + 2$ pendent vertices so that $\eta_{ia}(G') = n + 4$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G' . Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 5$. Hence, $\eta_{ia}(G) \leq n + 5$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and

atleast four vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 4$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 4 = n + 5$.

If $j = 3$, similar as above.

If $j = 4$, similar as above if no vertex of C_l except u_{l-1} is of $\text{deg} \geq 3$.

If $j = 4$ and each of C_l and C_m has a vertex other than u_{l-1} and u_{l+i-1} are of $\text{deg} \geq 3$. Let e be an edge in $D(l, m; i)$ not adjacent to either u_{l-1} or u_{l+i-1} , then $G' = G - e$ is a unicyclic graph with $n + 1$ pendent vertices so that $\eta_{ia}(G') = n + 2$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G' . Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast two vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

If $j = 5$, similar as above.

If $j = 6$. Let e be an edge in C_l not adjacent to u_{l-1} . Then $G_1 = G - e$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_{ia}(G) = q - p + t = n + 1$.

(ii). Case(a). If $G = D(l, m; i)$ then $\eta_{ia}(G) = 4$.

Case(b). When $j = 2$. Then $G_1 = G - C_m$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n + 2$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 4$. Hence, $\eta_{ia}(G) \leq n + 4$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast three vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 3$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 3 = n + 4$.

Similarly, we can prove for $j = 3$ and the third vertex of $\text{deg} \geq 3$ is adjacent to u_{l+i-1} in C_m .

Case(c). When $j = 3$ and the third vertex of $\text{deg} \geq 3$ is in C_l . Then $G_1 = G - C_m$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n + 1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast two vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Similarly, we can prove for $j \geq 3$ and all vertices of $\text{deg} \geq 3$ are in C_m by taking $G_1 = G - C_l$.

When $j = 4$ and v in C_l and w adjacent to u_{l+i-1} in C_m are of $\text{deg} \geq 3$. Let e be an edge in $D(l, m; i)$ not adjacent to u_{l-1} so that $G_1 = G - e$ is a unicyclic graph with $n + 1$ pendant vertices. Then $\eta_{ia}(G_1) = n + 2$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast two vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Case(d). When $j = 4$ and C_m has no vertex of $\text{deg} \geq 3$ other

than u_{l+i-1} . Then $G_1 = G - C_m$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 2$. Hence, $\eta_{ia}(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast one vertex in $D(l, m; i)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

When $j = 5$ and C_m has exactly one vertex of $\text{deg} \geq 3$ which is adjacent to u_{l+i-1} . Let e be an edge in C_l not adjacent to u_{l-1} . Then $G_1 = G - e$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n + 1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta_{ia}(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast one vertex in $D(l, m; i)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Case(e). When $j \geq 5$. Let e be an edge in $D(l, m; i)$ not incident to any vertex of P_i in C_l . Then $G_1 = G - e$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $D(l, m; i)$ and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_{ia}(G) = n + 1$.

(iii). Case(a). If $G = D(l, m; i)$ then $\eta_{ia}(G) = 3$.

Case(b). When $j = 2$. Then $G_1 = G - C_m$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n + 1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast two vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Similarly, we can prove for $j = 3$ and the third vertex of $\text{deg} \geq 3$ is adjacent to either u_{l-1} or u_{l+i-1} .

When $j = 4$ and from the vertices of $\text{deg} \geq 3$ other than u_{l-1} and u_{l+i-1} one is adjacent to u_{l-1} in C_l and other is adjacent to u_{l+i-1} in C_m . Suppose v with $\text{deg} \geq 3$ in C_m is adjacent to u_{l+i-1} . Let P be an induced path $v - u_{l+m+i-2}$ of length atleast two in C_m . Then $G_1 = G - P$ is a unicyclic graph with $n + 1$ pendant vertices so that $\eta_{ia}(G_1) = n + 2$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast two vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Case(c). When $j \geq 3$ and all vertices of $\text{deg} \geq 3$ are only in C_l or C_m only; or $j = 3$ and the other vertex of $\text{deg} \geq 3$ is adjacent to neither u_{l-1} nor u_{l+i-1} in G . Suppose v with $\text{deg} \geq 3$ lies in C_m . Then $G_1 = G - C_l$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup C_l$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 2$. Hence, $\eta_{ia}(G) \leq n + 2$. Again, for any induced acyclic

graphoidal cover ψ of G , the n pendant vertices of G and atleast one vertex in $D(l, m; i)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Case(d). Take two non adjacent vertices v_i and v_j of $\text{deg} \geq 3$ in C_l (or C_m). Let T be the induced subgraph of G containing all vertices on one side of the arc $v_i - v_j$ of C_l (or C_m) such that these two vertices appear as pendant vertices and T has $n_1 + 2$ pendant vertices. Then $\eta_{ia}(T_1) = n_1 + 1$. Let ψ_1 be the minimum induced acyclic graphoidal cover of T . Then $G_1 = G - T$ is a unicyclic graph with n_2 pendant vertices so that $n = n_1 + n_2$ and $\eta_{ia}(G_1) = n_2$. Let ψ_2 be the minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup \psi_2$, is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_{ia}(G) = n + 1$. ■

Remark II.11. In case P_i in $D(L, m; i)$ has any intermediate vertex(ices) of degree greater than or equal to 3 there will be no change in the minimum induced acyclic graphoidal covering number.

Theorem II.12. Let G be a bicyclic graph with n pendant vertices containing a $C_m(i; l)$ and j be the number of vertices of degree greater than or equal to 3 in $C_m(i; l)$. Then

(i) $\eta_{ia}(G) = 3$ if $G = C_m(i; l)$.

and when

(ii) $l = 1$

$$\eta_{ia}(G) = \begin{cases} n + 3 & \text{if } j = 2; \\ n + 2 & \text{if } j \geq 3 \text{ and all the vertices of } \text{deg} \geq 3 \\ & \text{are in one side of } P_l; \\ n + 1 & \text{otherwise.} \end{cases}$$

(iii) $l \geq 2$

$$\eta_{ia}(G) = \begin{cases} n + 2 & \text{if } \text{deg} u_0 = 3 \text{ and either } j = 2 \text{ or } j = 3 \text{ with} \\ & \text{the third vertex of } \text{deg} \geq 3 \text{ is adjacent to } u_i; \\ n + 1 & \text{otherwise.} \end{cases}$$

Proof: $G = C_m(i; l)$, so it contains atleast $C_m = \{u_0, u_1, \dots, u_i, u_{i+1}, \dots, u_{m-1}, u_0\}$ with $m \geq 4$ and the chord $P_l = \{u_0, u_m, u_{m+1}, \dots, u_{m+l-2}, u_i\}$, $l \geq 1$ and $2 \leq i \leq m - 2$.

(i). If $G = C_m(i; l)$ then $\eta_{ia}(G) = 3$.

(ii) Case(a). When $j = 2$. let u_s , $0 < s < i$, be any vertex in $C_m(i; l)$. Then $P_1 = (u_0, u_s)$, $P_2 = (u_s, u_{s+1}, \dots, u_i)$ be induced paths in $C_m(i; l)$. Let $G_1 = G - \{P_1, P_2\}$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n + 1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup P_1 \cup P_2$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 1 + 2$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast two vertices in $C_m(i; l)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Case(b). When $j \geq 3$ and all the vertices of $\text{deg} \geq 3$ are in one side of P_l , say (u_i, u_m, \dots, u_0) . Take a vertex u_s , $0 <$

$s < i$ in $C_m(i; l)$. Then the paths $P_1 = u_0 - u_s$ and $P_2 = u_s - u_i$ are induced paths in G . Let $G_1 = G - \{P_1, P_2\}$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup P_1 \cup P_2$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 2$. Hence, $\eta_{ia}(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast one vertex in $C_m(i; l)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Case(c). When $j \geq 4$, suppose u_s ($0 < s < i$) and u_t ($i < t < m - 1$) are two vertices of $\deg \geq 3$ in $C_m(i; l)$. Take the tree T of $C_m(i; l)$ on one side of P_l containing u_s such that u_0 and u_i are pendant vertices. Again, bifurcate T into two trees T_1 and T_2 containing $\{u_0, u_1, \dots, u_s\}$ and $\{u_s, u_{s+1}, \dots, u_i\}$ along with the subgraphs of T incident to these vertices such that $\deg u_s = 1$ in T_1 . Suppose T_1 and T_2 have $n_1 + 2$ and $n_2 + 1$ pendant vertices respectively. Then $\eta_{ia}(T_1) = n_1 + 1$ and $\eta_{ia}(T_2) = n_2$. Let ψ_1 and ψ_2 be respectively the minimum induced acyclic graphoidal cover of T_1 and T_2 . Also, $G' = G - T$ is a unicyclic graph with n_3 pendant vertices so that $n_1 + n_2 + n_3 = n$ and $\eta_{ia}(G') = n_3$. Let ψ_3 be a minimum induced acyclic graphoidal cover of G' . Then $\psi = \psi_1 \cup \psi_2 \cup \psi_3$ is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_{ia}(G) = n_1 + n_2 + n_3 + 1 = n + 1$.

(iii). Case(a). When $j = 2$ and $\deg u_0 = 3$.

Let $P = \{u_0 u_m u_{m+1} \dots u_{l+m-2} u_i\}$, $2 \leq i \leq m - 2$, be the chord in $C_m(i; l)$ such that $\eta_{ia}(P) = 1$. Then $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n + 1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta_{ia}(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover ψ of G , the n pendant vertices of G and atleast one vertex in $C_m(i; l)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Similarly, we can prove for $j = 3$ and the third vertex of $\deg \geq 3$ is adjacent to u_0 .

Case(b). Let $P = \{u_0 u_m u_{m+1} \dots u_{l+m-2} u_i\}$, $2 \leq i \leq m - 2$, be the chord in $C_m(i; l)$ such that $\eta_{ia}(P) = 1$. Then $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_{ia}(G) = n + 1$.

Otherwise, let T be the induced subgraph of G with vertex set $\{u_0 u_m u_{m+1} \dots u_{l+m-2} u_i\}$, $2 \leq i \leq m - 2$, along with vertices incident to this vertex set such that $\deg u_0, u_{l+m-2} = 1$. Then T has $n_1 + 2$ pendant vertices so that $\eta_{ia}(T) = n_1 + 1$. Let ψ_1 be the minimum induced acyclic graphoidal cover of T . Then $G_1 = G - T$ is a unicyclic graph with n_2 pendant vertices so that $n = n_1 + n_2$ and $\eta_{ia}(G_1) = n_2$. Let ψ_2 be the minimum induced acyclic graphoidal cover of G_1 . Then $\psi = \psi_1 \cup \psi_2$, is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ . Hence, $\eta_{ia}(G) = n + 1$. ■

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