

# Design of Nonlinear Observer by Using Chebyshev Interpolation based on Formal Linearization

Kazuo Komatsu and Hitoshi Takata

*Abstract*—This paper discusses a design of nonlinear observer by a formal linearization method using an application of Chebyshev Interpolation in order to facilitate processes for synthesizing a nonlinear observer and to improve the precision of linearization.

A dynamic nonlinear system is linearized with respect to a linearization function, and a measurement equation is transformed into an augmented linear one by the formal linearization method which is based on Chebyshev interpolation. To the linearized system, a linear estimation theory is applied and a nonlinear observer is derived. To show effectiveness of the observer design, numerical experiments are illustrated and they indicate that the design shows remarkable performances for nonlinear systems.

*Keywords*—nonlinear system, nonlinear observer, formal linearization, Chebyshev interpolation.

## I. INTRODUCTION

**N**ATURALLY, the estimation problem is more difficult and less understood when systems are nonlinear than linear. The most practical way to approach the nonlinear problem is to employ linearization in order to apply the linear system theories [1]–[6]. Formal linearization [7]–[10] is one of them to treat with these nonlinear problems.

In the previous work, a nonlinear observer design using the formal linearization method based on Chebyshev expansion was considered [10]. In this paper, we develop a nonlinear observer design by the formal linearization method based on Chebyshev interpolation in order to make processes of the design easier. Introducing a linearization function which consists of the Chebyshev polynomials of the state variables, and an augmented measurement vector which consists of polynomials of the measurement variables, a given nonlinear system is transformed into an augmented linear system by using Chebyshev interpolation. By this linearized system, a nonlinear observer is derived by applying linear system theories. Inversion is simple because of the original state variable involved in the linearization function.

An advantage in this method comparing with the previous work [10] is that coefficients of linearized system are simply obtained by carrying out summation due to the orthogonality for a finite sum, while they are executed by the integral calculus in the previous method.

Numerical experiments are illustrated to verify the effectiveness of this observer design in comparison with a conventional linearization based on Taylor expansion truncated at the first order.

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## II. STATEMENT OF PROBLEM

Consider a nonlinear dynamic system described by a state differential equation

$$\begin{aligned} \Sigma_1 : \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t)), \\ \mathbf{x}(0) &= \mathbf{x}_0 \in D \end{aligned} \quad (1)$$

where  $t$  denotes time,  $\dot{\cdot} = d/dt$ ,  $\mathbf{x}$  is an  $n \times 1$  state vector, and  $\mathbf{f}$  is a sufficiently smooth nonlinear function.  $D$  is a compact domain denoted by the Cartesian product:

$$D = \prod_{i=1}^n [l_i - p_i, l_i + p_i] \subset R^n$$

where  $l_i$  ( $l_i \in R$ ) is the middle of the domain of  $x_i$  and  $p_i$  ( $p_i > 0$ ) is half of the domain of  $x_i$  ( $i = 1, \dots, n$ ).

Assumed that a measurement equation is given by

$$\boldsymbol{\eta}(t) = \mathbf{h}(\mathbf{x}(t)) \in R^\ell \quad (2)$$

where  $\boldsymbol{\eta}$  is an  $\ell \times 1$  output vector with  $\ell < n$ , and  $\mathbf{h}(\mathbf{x})$  is a sufficiently smooth nonlinear function.

The problem is that the state of a nonlinear dynamic system (Eq. (1)) can be estimated by use of the given measurement output which is written by a nonlinear equation (Eq. (2)).

## III. NONLINEAR OBSERVER BY FORMAL LINEARIZATION

### A. Formal Linearization for Dynamic System

In this method, Chebyshev interpolation is applied to linearize the given nonlinear system (Eq. (1)), and the state variable  $\mathbf{x}$  is changed into  $\mathbf{y}$  so that  $\mathbf{y}$  has the basic domain of the Chebyshev polynomials

$$D_0 = \prod_{i=1}^n [-1, 1]$$

and  $\mathbf{y}$  is rewritten by

$$\mathbf{y} = P^{-1}(\mathbf{x} - L) \in D_0 \quad (3)$$

where

$$L = \begin{pmatrix} l_1 \\ \vdots \\ l_n \end{pmatrix}, P = \begin{pmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_n \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

The given dynamic system (Eq. (1)) becomes

$$\dot{\mathbf{y}}(t) = P^{-1} \mathbf{f}(P\mathbf{y}(t) + L). \quad (4)$$

The Chebyshev polynomials  $\{T_r(\cdot)\}$  are defined by

$$T_r(y_i) = \cos(r \cdot \cos^{-1} y_i), \quad (r = 0, 1, 2, \dots) \quad (5)$$

or,

$$T_0(y_i) = 1, T_1(y_i) = y_i, T_2(y_i) = 2y_i^2 - 1, \\ T_3(y_i) = 4y_i^3 - 3y_i, T_4(y_i) = 8y_i^4 - 8y_i^2 + 1, \dots$$

Its recurrence formula is

$$T_{q+1}(y_i) = 2y_i T_q(y_i) - T_{q-1}(y_i), (q \geq 1) \quad (6)$$

$$T_0(y_i) = 1, T_1(y_i) = y_i.$$

Therefore, the derivative of the Chebyshev polynomials

$$S_q(y_i) \equiv \frac{dT_q(y_i)}{dy_i} \quad (7)$$

is given by

$$S_{q+1}(y_i) = 2T_q(y_i) + 2y_i S_q(y_i) - S_{q-1}(y_i), (q \geq 1) \quad (8)$$

$$S_0(y_i) = 0, S_1(y_i) = 1.$$

Using these Chebyshev polynomials, we define an  $N$ -th order linearization function  $\phi(\cdot) = \phi(\mathbf{y}(\cdot))$  which consists of the Chebyshev polynomials by

$$\begin{aligned} \phi &= [\phi_1, \phi_2, \dots, \phi_i, \dots, \phi_{(N+1)^n-1}]^T \\ &= [T_{(10\dots0)}(\mathbf{y}), T_{(01\dots0)}(\mathbf{y}), \dots, T_{(0\dots01)}(\mathbf{y}), \\ &T_{(11\dots0)}(\mathbf{y}), T_{(101\dots0)}(\mathbf{y}), \dots, T_{(10\dots1)}(\mathbf{y}), \\ &T_{(20\dots0)}(\mathbf{y}), T_{(02\dots0)}(\mathbf{y}), \dots, T_{(r_1\dots r_n)}(\mathbf{y}), \\ &\dots, T_{(N\dots N)}(\mathbf{y})]^T \end{aligned} \quad (9)$$

where

$$T_{(r_1\dots r_n)}(\mathbf{y}) = \prod_{i=1}^n T_{r_i}(y_i).$$

The derivative of each element of  $\phi$  along with the solution of the given nonlinear system (Eq. (1)) becomes

$$\dot{\phi}_\alpha(\mathbf{y}) = \dot{T}_{(r_1\dots r_n)}(\mathbf{y}) = \frac{\partial T_{(r_1\dots r_n)}(\mathbf{y})}{\partial \mathbf{y}^T} \dot{\mathbf{y}}$$

$$= [S_{r_1}(y_1)T_{r_2}(y_2) \dots T_{r_{n-1}}(y_{n-1})T_{r_n}(y_n), \dots,$$

$$T_{r_1}(y_1)T_{r_2}(y_2) \dots T_{r_{n-1}}(y_{n-1})S_{r_n}(y_n)]P^{-1}\mathbf{f}(P\mathbf{y} + L) \\ \equiv G_{(r_1\dots r_n)}(\mathbf{y}), \alpha = \alpha(r_1, \dots, r_n). \quad (10)$$

Applying Chebyshev interpolation up to the  $N$ -th order, this  $G_{(r_1\dots r_n)}(\mathbf{y})$  is approximated by

$$\hat{G}_{(r_1\dots r_n)}(\mathbf{y}) = \sum_{q_1=0}^N \dots \sum_{q_n=0}^N C_{(q_1\dots q_n)}^{(r_1\dots r_n)} T_{(q_1\dots q_n)}(\mathbf{y}) \quad (11)$$

where

$$\begin{aligned} C_{(q_1\dots q_n)}^{(r_1\dots r_n)} &\equiv \frac{2^{n-\gamma}}{n} \sum_{j_1=0}^N \sum_{j_2=0}^N \dots \\ &\prod_{i=1}^n (N+1) \\ &\sum_{j_n=0}^N G_{(r_1\dots r_n)}(y_{1j_1}, y_{2j_2}, \dots, y_{nj_n}) \\ &\times T_{q_1}(y_{1j_1})T_{q_2}(y_{2j_2}) \dots T_{q_n}(y_{nj_n}), \end{aligned} \quad (12)$$

$$\gamma = \{\text{the number of } q_i = 0 : 1 \leq i \leq n\}.$$

The interpolating points  $\{y_{ij_i}\}$  are set to be

$$y_{ij_i} = \cos \frac{2j_i + 1}{2N + 2} \pi, (i = 1, \dots, n, j_i = 0, \dots, N). \quad (13)$$

Substituting this  $\hat{G}_{(r_1\dots r_n)}(\mathbf{y})$  into Eq. (10) yields

$$\dot{\phi}(\mathbf{y}) \approx A\phi(\mathbf{y}) + b \quad (14)$$

where

$$[A_\alpha \beta] = [C_{(q_1\dots q_n)}^{(r_1\dots r_n)}] \in R^{((N+1)^n-1) \times ((N+1)^n-1)},$$

$$[b_\alpha] = [C_{(0\dots0)}^{(r_1\dots r_n)}] \in R^{(N+1)^n-1}, \beta = \beta(q_1, \dots, q_n).$$

Thus a formal linear state differential equation is derived by

$$\Sigma_2 : \dot{\mathbf{z}}(t) = A\mathbf{z}(t) + b, \quad (15)$$

$$\mathbf{z}(0) = \phi(\mathbf{y}(0)) = \phi(P^{-1}(\mathbf{x}(0) - L)).$$

From Eqs. (3) and (9), the inversion is carried out as  $\hat{\mathbf{x}}(t)$  by

$$\begin{aligned} \hat{\mathbf{x}}(t) &= P[I \ 0 \ \dots \ 0]\phi(\mathbf{y}(t)) + L \\ &= P[I \ 0 \ \dots \ 0]\mathbf{z}(t) + L \end{aligned} \quad (16)$$

where  $I$  is an  $n \times n$  unit matrix.

### B. Formal Linearization for Measurement Equation

An augmented  $M$ -th order measurement vector  $\mathbf{Y}(\cdot) = \mathbf{Y}(\boldsymbol{\eta}(\cdot))$  which consists of polynomials of the measurement variables are defined as

$$\begin{aligned} \mathbf{Y} &= [Y_1, Y_2, \dots, Y_i, \dots, Y_{(M+1)^\ell-1}]^T \\ &= [T'_{(10\dots0)}(\boldsymbol{\eta}), T'_{(01\dots0)}(\boldsymbol{\eta}), \dots, T'_{(0\dots01)}(\boldsymbol{\eta}), \\ &T'_{(11\dots0)}(\boldsymbol{\eta}), T'_{(101\dots0)}(\boldsymbol{\eta}), \dots, T'_{(10\dots1)}(\boldsymbol{\eta}), \\ &T'_{(20\dots0)}(\boldsymbol{\eta}), T'_{(02\dots0)}(\boldsymbol{\eta}), \dots, T'_{(r_1\dots r_\ell)}(\boldsymbol{\eta}), \\ &\dots, T'_{(M\dots M)}(\boldsymbol{\eta})]^T \end{aligned} \quad (17)$$

where

$$T'_{(r_1\dots r_\ell)}(\boldsymbol{\eta}) = \prod_{i=1}^\ell \eta_i^{r_i}.$$

From the given measurement equation (Eq. (2)), each element function of Eq. (17) is written as

$$\begin{aligned} Y_{\alpha'}(\boldsymbol{\eta}) &= T'_{(r_1\dots r_\ell)}(\boldsymbol{\eta}) = \prod_{i=1}^\ell \eta_i^{r_i} \\ &= h_1^{r_1}(P\mathbf{y} + L)h_2^{r_2}(P\mathbf{y} + L) \dots h_\ell^{r_\ell}(P\mathbf{y} + L) \\ &\equiv G'_{(r_1\dots r_\ell)}(\mathbf{y}), \alpha' = \alpha'(r_1, \dots, r_\ell). \end{aligned} \quad (18)$$

To this new augmented measurement equation, apply Chebyshev interpolation up to the  $N$ -th order, and this  $G'_{(r_1\dots r_\ell)}(\mathbf{y})$  is approximated by

$$\hat{G}'_{(r_1\dots r_\ell)}(\mathbf{y}) = \sum_{q_1=0}^N \dots \sum_{q_n=0}^N C'_{(q_1\dots q_n)}^{(r_1\dots r_\ell)} T_{(q_1\dots q_n)}(\mathbf{y}) \quad (19)$$

where

$$C'_{(q_1\dots q_n)}^{(r_1\dots r_\ell)} \equiv \frac{2^{n-\gamma'}}{n} \sum_{j_1=0}^N \sum_{j_2=0}^N \dots \prod_{i=1}^n (N+1)$$

$$\sum_{j_n=0}^N G'_{(r_1 \dots r_\ell)}(y_{1j_1}, y_{2j_2}, \dots, y_{nj_n}) \times T_{q_1}(y_{1j_1}) T_{q_2}(y_{2j_2}) \dots T_{q_n}(y_{nj_n}), \quad (20)$$

$$\gamma' = \{\text{the number of } q_i = 0 : 1 \leq i \leq n\}.$$

Substituting this  $\hat{G}'_{(r_1 \dots r_\ell)}(\mathbf{y})$  into Eq.(18), the augmented measurement equation becomes

$$\mathbf{Y} \approx D\phi(\mathbf{y}) + e \quad (21)$$

where

$$[D_{\alpha'} \beta'] = [C'_{(q_1 \dots q_n)}] \in R^{((M+1)^\ell - 1) \times ((N+1)^n - 1)},$$

$$[e_{\alpha'}] = [C'_{(0 \dots 0)}] \in R^{((M+1)^\ell - 1)}, \beta' = \beta'(q_1, \dots, q_n).$$

Thus a formal linear measurement equation is derived by

$$\mathbf{Y}(t) = D\mathbf{z}(t) + e. \quad (22)$$

### C. Design of Nonlinear Observer

To the above linearized system (Eqs. (15) and (22)), a linear estimation theory is applied so that the identity observer [11] is synthesized as

$$\dot{\hat{\mathbf{z}}}(t) = A\hat{\mathbf{z}}(t) + b + K(t)(\mathbf{Y}(t) - (D\hat{\mathbf{z}}(t) + e)), \quad (23)$$

$$\hat{\mathbf{z}}(0) = \phi(\hat{\mathbf{y}}(0)) = \phi(P^{-1}(\hat{\mathbf{x}}(0) - L))$$

where  $\hat{\mathbf{x}}(0)$  is an initial value of the observer,  $K(t)$  is an observer gain as

$$K(t) = \frac{1}{2}R(t)D^T W(t) \in R^{((N+1)^n - 1) \times ((M+1)^\ell - 1)}.$$

$R(t)$  satisfies the matrix Riccati differential equation as

$$\dot{R}(t) = AR(t) + R(t)A^T + Q(t) - R(t)D^T W(t)DR(t) \quad (24)$$

where  $Q(t), W(t)$  and  $R(0)$  are chosen to be arbitrary real, symmetric, and positive definite. With the reference to the exponential estimator [11], the error in the state estimate  $e = z - \hat{z}$  is uniformly asymptotically stable in the sense of Lyapunov.

From Eq.(16), the estimate of the nonlinear observer  $\hat{\mathbf{x}}(t)$  becomes

$$\begin{aligned} \hat{\mathbf{x}}(t) &= P[I \ 0 \ \dots \ 0]\phi(\hat{\mathbf{y}}(t)) + L \\ &= P[I \ 0 \ \dots \ 0]\hat{\mathbf{z}}(t) + L. \end{aligned} \quad (25)$$

## IV. NUMERICAL EXPERIMENTS

Numerical experiments for nonlinear observers of scalar and multidimensional systems are illustrated. For comparison, a conventional linearization based on Taylor expansion truncated at the first order is also depicted.

### A. Nonlinear Observer for Scalar System

As a simple example, consider a dynamic scalar system

$$\dot{x} = x^2 \quad (26)$$

$$x(0) = 0.9, \quad D = [-1, 0] \subset R$$

and a measurement equation

$$\eta = \sqrt{1+x}. \quad (27)$$

To apply the above formal linearization in Sec. III, the values for changing state variable in Eq. (3) are set as

$$L = -0.5, P = 0.5,$$

the linearization function and the augmented measurement vector are

$$\phi = \begin{pmatrix} y \\ 2y^2 - 1 \\ 4y^3 - 3y \end{pmatrix}, \mathbf{Y} = \begin{pmatrix} \eta \\ \eta^2 \\ \eta^3 \end{pmatrix}$$

respectively, when the order of the linearization function  $N$  and the measurement vector  $M$  are  $N = M = 3$ . In this case, the formal linear system (Eq. (15)) becomes

$$\dot{\mathbf{z}}(t) = \begin{pmatrix} -1 & 0.25 & 0 \\ 3.5 & -2 & 0.5 \\ -6 & 5.25 & -3 \end{pmatrix} \mathbf{z}(t) + \begin{pmatrix} 0.75 \\ -2 \\ 3 \end{pmatrix},$$

and the augmented measurement equation (Eq. (22)) is

$$\mathbf{Y}(t) = \begin{pmatrix} 0.416 & -0.075 & 0.023 \\ 0.5 & 0 & 0 \\ 0.51 & 0.072 & -0.007221 \end{pmatrix} \mathbf{z}(t) + \begin{pmatrix} 0.641 \\ 0.5 \\ 0.424 \end{pmatrix}.$$

Fig. 1 shows the true value  $x$  of Eq. (26) and the approximated values  $\hat{x}$  of Eq.(16) when  $N$  is varied from  $N = 1$  to 3.  $\hat{x}(\text{Taylor})$  refers to a result obtained by Taylor expansion truncated at the first order :

$$\dot{\hat{x}} = -\hat{x} - \frac{1}{4}$$

when the operating point is  $-0.5$  for comparison.

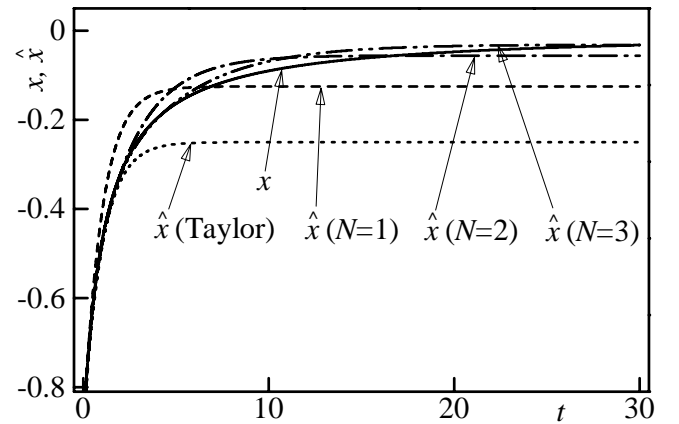


Fig. 1. True value  $x(t)$  and approximated values  $\hat{x}(t)$  of linearization

Fig. 2 depicts the true value  $\eta$  of Eq. (27) and the approximated values  $\mathbf{Y}$  of Eq.(21) when  $N$  is varied from  $N = 1$  to

3 and  $M$  is fixed at 1.  $\hat{\eta}$ (Taylor) is a result obtained by Taylor expansion truncated at the first order :

$$\hat{\eta} = \frac{1}{\sqrt{2}}x + \frac{3\sqrt{2}}{4}$$

when the operating point is  $-0.5$ .

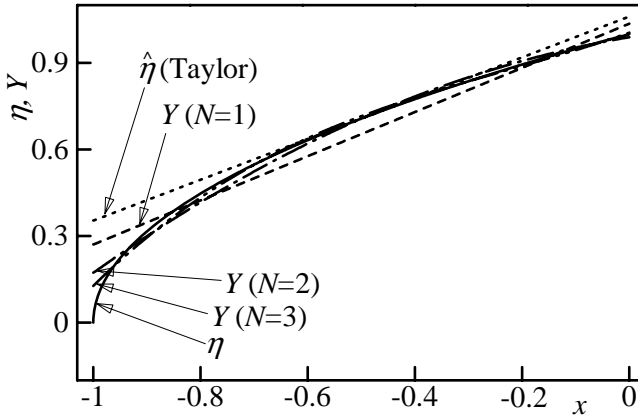


Fig. 2. True value  $\eta$  and approximated values of measurement equation

To synthesize the nonlinear observer for the given system (Eqs. (26) and (27)), parameters for the nonlinear observer (Eq. (23)) are set by the unknown value  $x(0) = 0.9$ , an initial value of the observer  $\hat{x}(0) = -0.01$  in Eq. (23),

$$R(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{diag}(1, 1, 1),$$

$$W(t) = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \text{diag}(5, 5, 5)$$

when  $N = 3$ ,

$$R(0) = \text{diag}(1, 1), W(t) = \text{diag}(5, 5)$$

when  $N = 2$ , and

$$R(0) = 1, W(t) = 5$$

when  $N = 1$ .

Fig. 3 shows true value  $x$  and estimates  $\hat{x}$  of Eq. (25) when the orders of  $N$  and  $M$  are varied from 1 to 3.  $\hat{x}$  (Taylor) is a result by the conventional first order Taylor expansion for comparison when the parameters for the observer are the same  $R(0) = 1$  and  $W(t) = 5$  as our method of the order  $N = M = 1$ .

Fig. 4 shows the integral square errors of estimation

$$J(t) = \int_0^t (x(\tau) - \hat{x}(\tau))^2 d\tau$$

for the various orders from  $N = M = 1$  to  $N = M = 3$  and the conventional Taylor method(Taylor).

Next experimentats are the results when the order of linearization function is fixed at  $N = 3$  and the order of the augmented measurement vector is varied from  $M = 1$  to 3. Fig. 5 shows  $x$  and  $\hat{x}$  . Fig. 6 shows the integral square errors of estimation in this case.

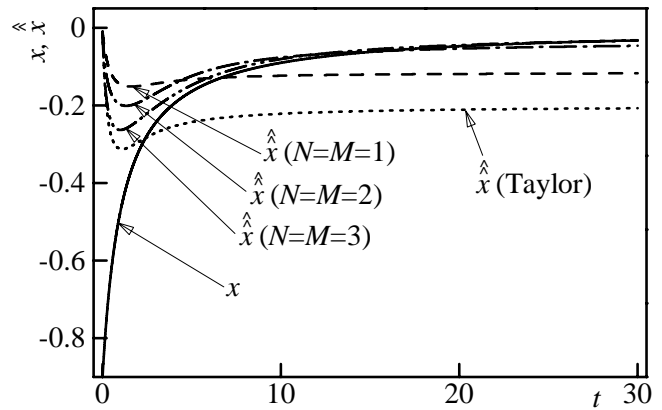


Fig. 3. Estimates  $\hat{x}(t)$  of the scalar system by various orders

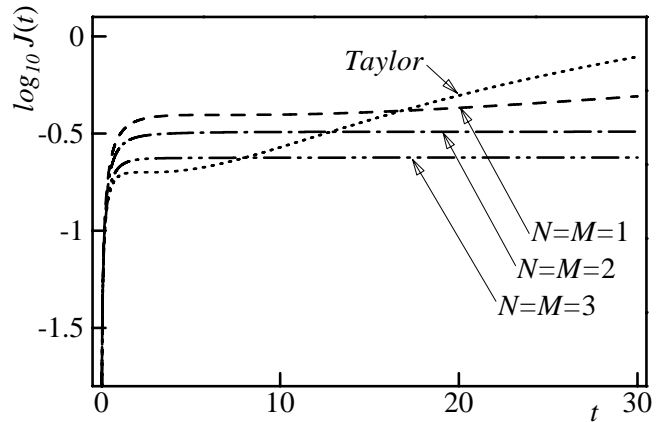


Fig. 4. Integral square errors of estimation of the scalar system by various orders

### B. Nonlinear Observer for Multidimensional System

As a multidimensional system, consider the simple pendulum in which the bob is connected to the rod which has zero mass. Let  $\theta$  denote the angle subtended by the rod and the vertical axis through the pivot point. The dynamic equation of this system is written as

$$\frac{d^2}{dt^2}\theta + a_1 \frac{d}{dt}\theta + a_2 \sin(\theta) = 0. \quad (28)$$

Assume that the position of the bob is measured from above and the measurement equation is

$$\eta = a_3 \sin(\theta). \quad (29)$$

Taking the state variables as  $x_1 = \theta$  and  $x_2 = \dot{\theta}$ , the dynamic system (Eq. (28)) is described as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a_1 x_2 - a_2 \sin(x_1) \end{cases} \quad (30)$$

and the measurement equation is

$$\eta = a_3 \sin(x_1) \quad (31)$$

Applying the above formal linearization in Sec. III, a formal linear system is obtained by

$$\dot{z}(t) = Az(t) + b, \quad (32)$$

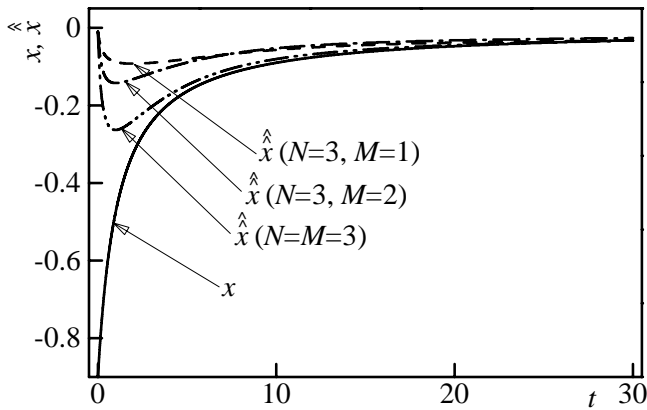


Fig. 5. Estimates  $\hat{x}(t)$  of the scalar system by various orders of  $M$  when  $N$  is fixed at 3

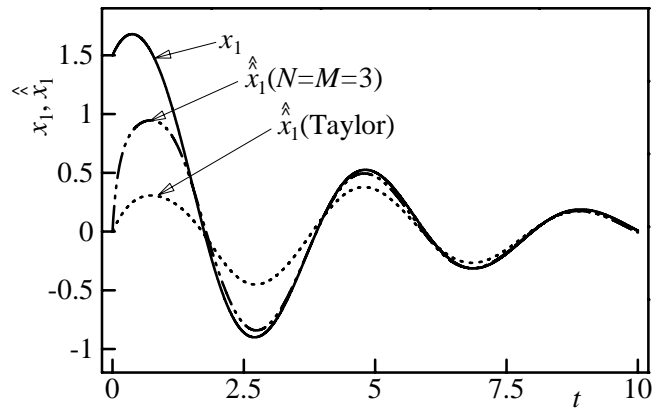


Fig. 7. Estimates  $\hat{x}_1(t)$  of the pendulum system by new method and the conventional method

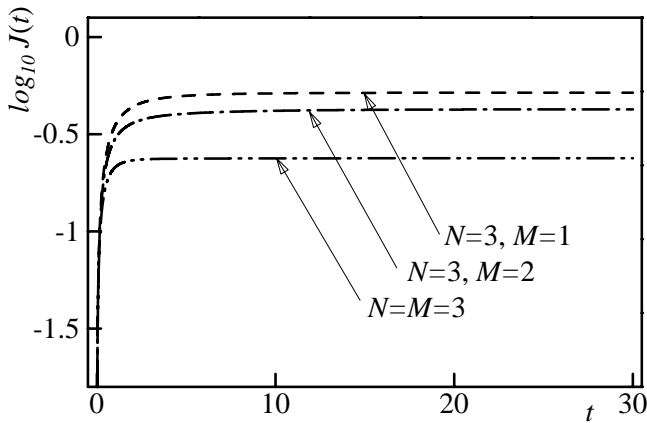


Fig. 6. Integral square errors of estimation of the scalar system by various orders of  $M$  when  $N$  is fixed at 3

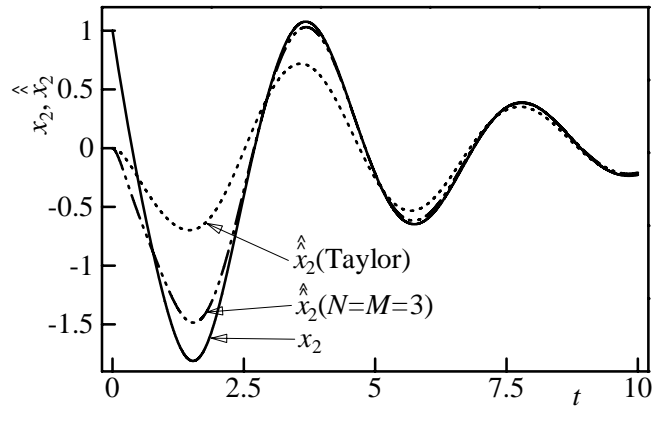


Fig. 8. Estimates  $\hat{x}_2(t)$  of the pendulum system by new method and the conventional method

$$Y(t) = Cz(t) + d, \tag{33}$$

and the observer of this pendulum system is

$$\begin{aligned} \dot{\hat{z}}(t) &= A\hat{z}(t) + b + K(t)(Y(t) - C\hat{z}(t) - d), \tag{34} \\ \hat{z}(0) &= \phi(\hat{x}(0)). \end{aligned}$$

Throughout this experiments, the system parameters are set as

$$a_1 = 0.5, a_2 = \frac{980.7}{400}, a_3 = 1.$$

For the formal linearization, the values for changing state variable in Eq. (3) are set as

$$L = \begin{pmatrix} 0.4 \\ -0.4 \end{pmatrix}, P = \begin{pmatrix} 1.3 & 0 \\ 0 & 1.5 \end{pmatrix}.$$

To synthesize the observer, parameters for the nonlinear observer (Eq. (34)) are set by

$$x(0) = \begin{pmatrix} 1.5 \\ 1 \end{pmatrix}, \hat{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$R(0) = I, W(t) = \text{diag}(10, 5, 2), N = 3, \text{ and } M = 3.$$

Figs. 7 and 8 show the true values  $x_i(t)$  and the estimated values  $\hat{x}_i(t)$  for  $i = 1$  and 2, respectively.  $\hat{x}_i(\text{Taylor})$  are the

results synthesized by the conventional observer based on the linearization of Taylor expansion truncated at the first order whose system is written by

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 \\ \dot{\hat{x}}_2 = -a_1 \hat{x}_2 - a_2 \hat{x}_1 \end{cases}, \quad \hat{\eta} = a_3 \hat{x}_1$$

when the operating point is the origin for comparison.

Fig. 9 shows the integral square errors of estimation

$$J(t) = \int_0^t (x(\tau) - \hat{x}(\tau))^T (x(\tau) - \hat{x}(\tau)) d\tau$$

for the various orders of the linearization function  $N$  and the augmented measurement vector  $M$  from 1 to 3. *Taylor* is the error by the conventional method based on Taylor expansion. Fig. 10 shows the errors of estimation for the various orders of  $M$  from 1 to 3 when the order of  $N$  is fixed at 3.

### V. CONCLUSIONS

We have developed an observer design for a nonlinear system by a formal linearization method exploiting Chebyshev interpolation. By this method, a nonlinear observer design is

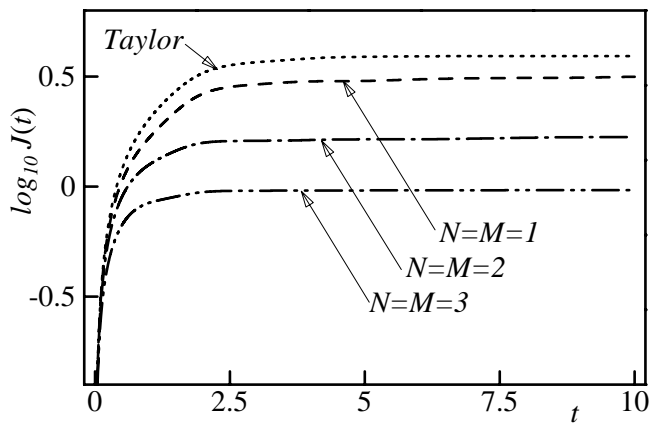


Fig. 9. Integral square errors of estimation of the pendulum system by various orders

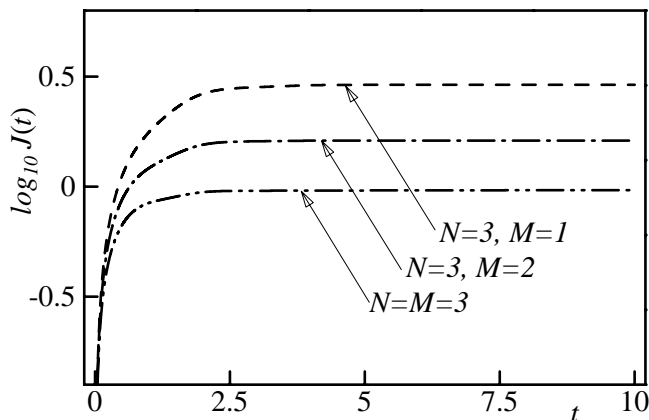


Fig. 10. Integral square errors of estimation of the pendulum system by various orders of  $M$  when  $N$  is fixed at 3

synthesized easier than the previous work because linearization processes are mechanically computed with a computer.

Numerical experiments show that our method is better than the previous works and the accuracy is improved as both the orders of the linearization function and the augmented measurement vector increase.

## REFERENCES

- [1] A. J. Krener, "Approximate Linearization by State Feedback and Coordinate Change," *Systems and Control Letters*, Vol.5, pp.181–185, 1984.
- [2] R. Marino, "On the Largest Feedback Linearizable Subsystem," *Systems and Control Letters*, Vol.6, pp.345–351, 1986.
- [3] W. T. Baumann and W. J. Rugh, "Feedback Control of Nonlinear Systems by Extended Linearization," *IEEE Trans.*, AC-31, 1, pp.40–46, 1986.
- [4] R. R. Kadiyala, "A Tool Box for Approximate Linearization on Nonlinear Systems," *IEEE Control Systems*, Vol.13, No.2, pp.47–57, 1993.
- [5] A. Ishidori, *Nonlinear Control Systems II*, Springer-Verlag, London, 1999.
- [6] H. K. Khalil, *Nonlinear Systems*, 3rd ed., New Jersey: Prentice Hall, 2002.
- [7] H. Takata, "Transformation of a Nonlinear System into an Augmented Linear System," *IEEE Trans. on Automatic Control*, Vol.AC-24, No.5, pp.736–741, 1979.
- [8] H. Takata, K. Komatsu and H. Sano, "Formal Linearization of Nonlinear Time-Varying Dynamic Systems Using Chebyshev and Laguerre Polynomials," *Proc. of the 16th IFAC World Congress*, pp.1–6, 2005.

- [9] K. Komatsu and H. Takata, "Design of Nonlinear Observer by Using Augmented Linear System based on Formal Linearization of Polynomial Type," *Proc. of World Academy of Science, Engineering and Technology*, pp.29–32, 2009.
- [10] K. Komatsu and H. Takata, "On a Nonlinear Observer via Formal Linearization based on Chebyshev Expansion," *Proc. of 2010 RISP International Workshop on Nonlinear Circuits, Communications and Signal Processing*, Hawaii, pp.576–579, 2010.
- [11] G. W. Johnson, "A Deterministic Theory of Estimation and Control," *IEEE Trans. on Automatic Control*, pp. 380–384, 1969.
- [12] A. P. Sage and C. C. White III, *Optimum Systems Control*, 2nd ed., Prentice-Hall, Inc., 1977.

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