On Certain Estimates Of Rough Oscillatory Singular Integrals

H. M. Al-Qassem

Abstract—We obtain appropriate sharp estimates for rough oscillatory integrals with polynomial phase. Our results represent significant improvements as well as natural extensions of what was known previously.

Keywords—Fourier transform, oscillatory integrals, Orlicz spaces, Block spaces, Extrapolation, L^p boundedness.

I. INTRODUCTION AND MAIN RESULTS

THROUGHOUT this paper, let $\mathbf{R}^n$, $n \geq 2$, be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue surface measure $d\sigma$.

Let K_Ω be a kernel of Calderön-Zygmund type on $\mathbf{R}^n$ given by

$$K_\Omega(x) = \Omega(x/|x|)\,|x|^{-n},$$

where Ω is a function defined on $\mathbf{S}^{n-1}$, integrable over $\mathbf{S}^{n-1}$ and satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(x)\, d\sigma(x) = 0. \tag{1}$$

Let $\mathcal{P}(n;m)$ denote the set of polynomials on $\mathbf{R}^n$ which have real coefficients and degrees not exceeding m, and let $\mathcal{H}(n;m)$ denote the collection of polynomials in $\mathcal{P}(n;m)$ which are homogeneous of degree m. For $P(x) = \sum_{|\eta|\leq m} a_\eta x^\eta$, we set $\|P\| = \sum_{|\eta|\leq m} |a_\eta|$. Let $n \geq 2$, $m \in \mathbf{N}$ and $\alpha > 0$. An integrable function Ω on $\mathbf{S}^{n-1}$ is said to be in the space $A(n;m;\alpha)$ if

$$\sup_{P\in\mathcal{H}(n;m),\|P\|=1} \int_{\mathbf{S}^{n-1}} |\Omega(y)| \left(\log \frac{1}{|P(y)|}\right)^{1+\alpha} d\sigma(y) < \infty. \tag{2}$$

For $\alpha \geq 0$, let $\mathbf{F}_\alpha(\mathbf{S}^{n-1})$ denote the space of all integrable functions Ω on $\mathbf{S}^{n-1}$ which satisfy the condition

$$\|\Omega\|_{\mathbf{F}_\alpha(\mathbf{S}^{n-1})} = \sup_{\xi\in\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} |\Omega(y)| \left(\log \frac{1}{|\xi\cdot y|}\right)^{1+\alpha} d\sigma(y) < \infty. \tag{3}$$

We point out the space $\mathbf{F}_\alpha(\mathbf{S}^{n-1})$ (with $\alpha > 0$) was introduced by Grafakos and Stefanov in [7] with respect to their studies of singular integrable operators. Also, it should be noted that Grafakos and Stefanov in [7] showed that for any $\alpha > 0$

$$\bigcup_{q>1} L^q(\mathbf{S}^{n-1}) \subsetneq \mathbf{F}_\alpha(\mathbf{S}^{n-1}), \tag{4}$$

$$\bigcap_{\alpha>0} \mathbf{F}_\alpha(\mathbf{S}^{n-1}) \nsubseteq H^1(\mathbf{S}^{n-1}) \nsubseteq \bigcup_{\alpha>0} \mathbf{F}_\alpha(\mathbf{S}^{n-1}), \tag{5}$$

where $H^1(\mathbf{S}^{n-1})$ denotes the Hardy space on $\mathbf{S}^{n-1}$ in the sense of Coifman and Weiss [6].

It was noted in [1] that $A(n;1;\alpha) = \mathbf{F}_\alpha(\mathbf{S}^{n-1})$ and in the case $n = 2$,

$$\bigcap_{m=1} A(2;m;\alpha) = \mathbf{F}_\alpha(\mathbf{S}^1). \tag{6}$$

However, $\mathbf{F}_\alpha(\mathbf{S}^{n-1}) \nsubseteq \mathbf{A}(n,m,\alpha)$ for $n \geq 3$.

Consider the oscillatory singular integral $I_\Omega(P)$ given by

$$I_\Omega(P) = \text{p.v.} \int_{\mathbf{R}^n} e^{iP(x)} K_\Omega(x)dx \text{ for } P \in \mathcal{P}(n;d). \tag{7}$$

One of the main issues of concern regarding these oscillatory singular integrals is obtaining sharp estimates for these integrals with constants depending only the degree of the polynomial P and also on a sharp size condition on Ω. The study of these problems was initiated by Stein-Wainger in [11], Stein in [10], and recently continued by Parissis in [8] and by Papadimtrakis-Parissis in [9].

In [10], Stein studied the singular integral $I_\Omega(P)$ and proved the following:

Theorem A. *Assume that $\Omega \in L^\infty(\mathbf{S}^{n-1})$ and satisfies* (1). *Then for any $P \in \mathcal{P}(n;d)$, there exists a positive constant c_d depending only on the degree d of the polynomial P and it is independent of its coefficients such that*

$$|I_\Omega(P)| \leq c_d \|\Omega\|_{L^\infty(\mathbf{S}^{n-1})}. \tag{8}$$

Recently, motivated by a result of Parissis in [8], Papadimtrakis and Parissis in [9] improved Stein's result by showing that the constant c_d can be replaced by $c(\log d)$ for some absolute constant c and that the condition on Ω can be weaken to be $\Omega \in L\log L(\mathbf{S}^{n-1})$. Their result can be stated as follows.

Theorem B. *Assume that $\Omega \in L\log L(\mathbf{S}^{n-1})$ and satisfies* (1). *Then there exists an absolute positive constant c such that*

$$\sup_{P\in\mathcal{P}(n;d)} |I_\Omega(P)| \leq c(\log d + 1)\left(1 + \|\Omega\|_{L\log L(\mathbf{S}^{n-1})}\right). \tag{9}$$

Recently, Al-Qassem et al. in [3] were able to show that Theorem C continues to hold if the condition $\Omega \in L\log L(\mathbf{S}^{n-1})$ is replaced by the weaker condition $\Omega \in H^1(\mathbf{S}^{n-1})$. It is worth mentioning that by Theorem A, one can easily show that if Ω is an odd function on $\mathbf{S}^{n-1}$ and Ω merely in $L^1(\mathbf{S}^{n-1})$, then

$$\sup_{P\in\mathcal{P}(n;d)} |I_\Omega(P)| \leq c(\log d + 1)\|\Omega\|_{L^1(\mathbf{S}^{n-1})}.$$

In light of the estimates in (8)-(9) and the inclusion relations in (4), the following question arises naturally:

Question. *Does an estimate of the form* (9) *holds under the condition $\Omega \in \mathbf{F}_\alpha(\mathbf{S}^{n-1})$ for some $\alpha \geq 0$.*

H. M. Al-Qassem is with the Department of Mathematics and Physics, Qatar University, Doha, Qatar, email: husseink@qu.edu.qa

The main purpose of this paper is to have an answer to the above question. The exact statements of our results are the following:

Theorem 1.1. *Let $n \geq 2$, $d \in \mathbf{N}$. Let Ω satisfy (1) and $\Omega \in \bigcap_{m=1} A(n;m;\alpha)$ for some $\alpha > 0$. Then there exists an absolute positive constant c which depends on Ω such that*

$$\sup_{P\in\mathcal{P}(n;d)} |I_\Omega(P)| \leq c(\log d + 1)\left(1 + C(\Omega)\right), \qquad (10)$$

where $C(\Omega)$ is a constant depends on Ω.

Corollary 1.2. *Let $n = 2$, $d \in \mathbf{N}$. Let Ω satisfy* (1) *and $\Omega \in \mathbf{F}_\alpha(\mathbf{S}^1)$ for some $\alpha > 0$. Then there exists an absolute positive constant c* such that

$$\sup_{P\in\mathcal{P}(n;d)} |I_\Omega(P)| \leq C(\log d + 1)\left(1 + \|\Omega\|_{\mathbf{F}_\alpha(\mathbf{S}^1)}\right). \qquad (11)$$

If $P \in \mathcal{P}(n;d)$ with $d = 1$, we have the following sharper result:

Theorem 1.2. *Let $n \geq 2$, $d = 1$. Let Ω satisfy* (1) *and $\Omega \in \mathbf{F}_0(\mathbf{S}^{n-1})$. Then there exists an absolute positive constant c which depends on Ω such that*

$$\sup_{P\in\mathcal{P}(n;1)} |I_\Omega(P)| \leq c(\|\Omega\|_{\mathbf{F}_0(\mathbf{S}^{n-1})} + \|\Omega\|_{\mathbf{L}^1(\mathbf{S}^{n-1})}). \qquad (12)$$

Throughout the rest of the paper, we always use the letter C to denote a positive constant that may vary at each occurrence but it is independent of the essential variables.

II. PROOF OF THEOREMS

Let first start with proving Theorem 1.2.

Proof of Theorem 1.2. Assume Ω satisfies (1) and $\Omega \in \mathbf{F}_0(\mathbf{S}^{n-1})$. Let $P \in \mathcal{P}(n;1)$. Without loss of generality, we may assume P does not have a constant term. Thus P is a polynomial given by $P_a(x) = a\cdot x$, where $a = (a_1,\ldots,a_n) \in \mathbf{R}^n$ and $x = (x_1,\ldots,x_n) \in \mathbf{R}^n$. By a change of variable we have

$$\begin{aligned} I_\Omega(P) &= \text{p.v.}\int_{\mathbf{R}^n} e^{iP(x)}K_\Omega(x)dx \\ &= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{\varepsilon\leq|x|\leq R} e^{iP(x)}\frac{\Omega(x/|x|)}{|x|^n}dx \\ &= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{\mathbf{S}^{n-1}}\int_{\varepsilon|a|}^{R|a|} e^{-i2\pi t\left(a'\cdot x\right)}\Omega(x)\frac{dt}{t}d\sigma(x), \end{aligned}$$

where $a' = a/|a|$ with $\mathbf{R}^n\backslash\{0\}$.

Since

$$\begin{aligned} &\int_\varepsilon^R \left(e^{-2\pi it\left(a'\cdot x\right)} - \cos(2\pi t)\right)\frac{dt}{t} \\ \to\ & \log|a'\cdot x|^{-1} - i\frac{\pi}{2}sgn(a'\cdot x) \end{aligned}$$

as $R\to\infty$ and $\varepsilon\to 0$, the integral is bounded, uniformly in ε and R, by $C\left(1 + |\log|a'\cdot x||\right)$.

Thus, using (1) and Lebesgue's dominated convergence theorem, we get

$$\begin{aligned} &I_\Omega(P) \\ =\ &\int_{\mathbf{S}^{n-1}} \Omega(x)\left(\log|a'\cdot x|^{-1} - i\frac{\pi}{2}sgn(a'\cdot x)\right)d\sigma(x). \end{aligned}$$

Therefore,

$$|I_\Omega(P)| \leq c(\|\Omega\|_{\mathbf{F}_0(\mathbf{S}^{n-1})} + \|\Omega\|_{\mathbf{L}^1(\mathbf{S}^{n-1})})$$

which completes the proof of Theorem 1.2.

Proof of Theorem 1.1. Assume that $\Omega \in \bigcap_{m=1} A(n;m;\alpha)$ for some $\alpha > 0$ and satisfies (1). Let

$$A_d = A_d(\Omega,n) = \sup_{\substack{0<\varepsilon<R,\\ P\in\mathcal{P}(n;d)}} |I_{\varepsilon,R}(P)|,$$

where

$$I_{\varepsilon,R}(P) = \int_{\varepsilon\leq|x|\leq R} e^{iP(x)}\frac{\Omega(x/|x|)}{|x|^n}dx.$$

We need to show that

$$|A_d(\Omega,n)| \leq C(\log d + 1)C(\Omega) \qquad (13)$$

for some absolute positive constant c and for some constant $C(\Omega)$ depends only on Ω. We shall first prove (13) for the case $d = 2^m$ for some integer $m \geq 0$ and then the general case will be an immediate consequence.

Switching to polar coordinates we get

$$I_{\varepsilon,R}(P) = \int_{\mathbf{S}^{n-1}}\int_\varepsilon^R e^{iP(tx)}\Omega(x)\frac{dt}{t}d\sigma(x).$$

We may assume without loss of generality that $P(tx)$ does not have a constant term. Write $P(tx) = \sum_{s=1}^d P_s(x)t^s$, where P_s is a homogeneous function of degree s. Let $m_j = \|P_j\|_{L^\infty(\mathbf{S}^{n-1})}$ and $Q(tx) = \sum_{s=1}^{d/2} P_s(x)t^s$. Since ϵ and R are arbitrary positive numbers and P is a polynomial of degree d, by a dilation in t we may assume, without loss of generality, that $\max_{\frac{d}{2}<j\leq d} m_j = 1$. Also, there is $\frac{d}{2} < j_0 \leq d$ so that $m_{j_0} = 1$. Now, $I_{\varepsilon,R}(P)$ can be written as

$$\begin{aligned} &|I_{\varepsilon,R}(P)| \\ \leq\ &\left|\int_{\mathbf{S}^{n-1}}\int_\varepsilon^1 e^{iP(tx)}\Omega(x)\frac{dt}{t}d\sigma(x)\right| \\ &+\left|\int_{\mathbf{S}^{n-1}}\int_1^R e^{iP(tx)}\Omega(x)\frac{dt}{t}d\sigma(x)\right| \\ =\ &I_1 + I_2. \end{aligned} \qquad (14)$$

Let us first estimate I_1 as follows:

$$\begin{aligned} &I_1 \\ \leq\ &\int_{\mathbf{S}^{n-1}}\int_0^1 \left|e^{iP(tx)} - e^{iQ(tx)}\right||\Omega(x)|\frac{dt}{t}d\sigma(x) \\ &+\left|\int_{\mathbf{S}^{n-1}}\int_\varepsilon^1 e^{iQ(tx)}\Omega(x)\frac{dt}{t}d\sigma(x)\right| \\ \leq\ &\sum_{\frac{d}{2}<j\leq d}\frac{m_j}{j}\|\Omega\|_{L^1(\mathbf{S}^{n-1})} + A_{d/2}. \end{aligned}$$

Therefore we have

$$I_1 \leq \|\Omega\|_{L^1(\mathbf{S}^{n-1})} + A_{d/2}. \qquad (15)$$

Now we estimate

$$I_2 = \left|\int_{\mathbf{S}^{n-1}}\int_1^R e^{iP(tx)}\Omega(x)\frac{dt}{t}d\sigma(x)\right|.$$

For each fixed $R > 1$ we have a unique $k_0 \in \mathbf{Z}_+$ such that $2^{k_0-1} \leq R < 2^{k_0}$. Hence

$$\begin{aligned} & I_2 \\ \leq\ & \sup_{k_0\in\mathbf{Z}_+}\left|\int_{\mathbf{S}^{n-1}}\int_{2^{k_0-1}}^{2^{k_0}} e^{iP(tx)}\Omega(x)\frac{dt}{t}d\sigma(x)\right| \\ & +\sup_{k_0\in\mathbf{Z}_+}\left|\sum_{k=k_0+1}^{\infty}\int_{\mathbf{S}^{n-1}}\int_{2^{k-1}}^{2^{k}} e^{iP(tx)}\Omega(x)\frac{dt}{t}d\sigma(x)\right| \\ =\ & J_1+J_2. \end{aligned} \tag{16}$$

Now we need the following lemma from [2].

Lemma. *Let $h(t) = b_0 + b_1 t + \cdots + b_d t^d$ be a real polynomial of degree at most d, and let $\psi \in \mathbf{C}^1[a, b]$. Then for any j_0 with $1 \leq j_0 \leq d$, there exists a positive constant C independent of a, b, the coefficients of $b_0, \ldots, b_d$ and also independent of d such that*

$$\begin{aligned} & \left|\int_a^b e^{ih(t)}\psi(t)dt\right| \\ \leq\ & C\left|b_{j_0}\right|^{-\frac{1}{d}}\left\{\sup_{a\leq t\leq b}|\psi(t)|+\int_a^b|\psi'(t)|\,dt\right\} \end{aligned}$$

holds for $0 < a < b \leq 1$.

It is easy to see that

$$J_1 \leq c\,\|\Omega\|_{L^1(\mathbf{S}^{n-1})}. \tag{17}$$

By the above lemma we get

$$\left|\int_{2^{-1}}^{1} e^{iP(2^k tx)}\frac{dt}{t}\right| \leq C\left|2^{j_0 k}P_{j_0}(x)\right|^{-\frac{1}{d}}.$$

By combining the last estimate with the trivial estimate

$$\left|\int_{2^{-1}}^{1} e^{iP(2^k tx)}\frac{dt}{t}\right| \leq \log 2,$$

we obtain

$$\begin{aligned} & \left|\int_{2^{-1}}^{1} e^{iP(2^k tx)}\frac{dt}{t}\right| \\ \leq\ & C\left(\log 2^{j_0 k}\right)^{-(\alpha+1)}\left(d+\alpha+\log\frac{1}{|P_{j_0}(x)|}\right)^{\alpha+1}. \end{aligned}$$

By the last inequality and since

$$(a+b)^{\theta} \leq 2^{\theta-1}\left(a^{\theta}+b^{\theta}\right)\ \text{(for } \theta\geq 1 \text{ and } a,b\geq 0)$$

we get

$$\begin{aligned} & \left|\int_{2^{-1}}^{1} e^{iP(2^k tx)}\frac{dt}{t}\right| \\ \leq\ & C\,(j_0 k)^{-(\alpha+1)}(d+\alpha)^{\alpha+1}\left(\log\frac{1}{|P_{j_0}(x)|}\right)^{\alpha+1} \\ \leq\ & Ck^{-(\alpha+1)}\left(\log\frac{1}{|P_{j_0}(x)|}\right)^{\alpha+1}. \end{aligned} \tag{18}$$

Therefore, by a change of variable, (18) and since $P_{j_0} \in \mathcal{H}(n; m)$ with $\|P_{j_0}\| = 1$, we get

$$J_2 \leq C\sup_{k_0\in\mathbf{Z}_+}\left(\sum_{k=k_0+1}^{\infty}k^{-(\alpha+1)}\right).$$

which in turn implies

$$J_2 \leq C. \tag{19}$$

By (16)–(17) and (19) we obtain

$$I_2 \leq C. \tag{20}$$

Thus by) (14), (15) and (20) we get

$$A_d \leq C + A_{d/2}.$$

Since $d = 2^m$, we get

$$A_{2^m} \leq C + A_{2^{m-1}}$$

and hence by induction on m we have

$$A_{2^m} \leq Cm + A_1. \tag{21}$$

Now, we need to estimate A_1. To this end, we notice that any $P \in \mathcal{P}(n; 1)$ with a non constant term will be of the form $P(x) = a.x$ for some $a \in \mathbf{R}^n$. By the calculations as in the proof of Theorem 1.2 and a change of variable we get

$$\begin{aligned} & |I_{\varepsilon,R}(P)| \\ \leq\ & \left|\int_{\mathbf{S}^{n-1}}\Omega(x)\left(\log|a'\cdot x|^{-1}-i\frac{\pi}{2}sgn(a'\cdot x)\right)d\sigma(x)\right|, \end{aligned}$$

where $a' = a/|a|$. Hence,

$$|I_{\varepsilon,R}(P)| \leq C + \left|\int_{\mathbf{S}^{n-1}}\Omega(x)\log|a'\cdot x|^{-1}d\sigma(x)\right|. \tag{22}$$

Since $\Omega \in \bigcap_{m=1} A(n; m; \alpha)$ we obtain $\Omega \in A(n; 1; \alpha) = \mathbf{F}_{\alpha}(\mathbf{S}^{n-1})$ which easily implies that

$$|I_{\varepsilon,R}(P)| \leq C,$$

and hence we have

$$A_1 \leq C. \tag{23}$$

Hence, by (19) and (22) we obtain

$$A_{2^m} \leq C(m+1). \tag{24}$$

The case now for the general d is easy. Choose a positive integer m so that $2^{m-1} < d \leq 2^m$. By definition of A_d and since $\mathcal{P}(n; d) \subset \mathcal{P}(n; 2^m)$ we have

$$A_d \leq A_{2^m} \leq C(m+1) \leq C(\log d + 1),$$

which completes the proof of Theorem 1.1.

References

[1] A. Al-Salman and Y. Pan, Singular integrals with rough kernels, Canad. Math. Bull. 47 (2004), no. 1, 3–11.

[2] H. M. Al-Qassem, L. Cheng, and Y. Pan, On the boundedness of rough oscillatory singular integrals on Trieble-Lizorkin spaces. To appear in Acta Mathematica Sinica.

[3] H. M. Al-Qassem, L. Cheng, A. Fukui and Y. Pan, Bounds for Oscillatory Singular Integrals on $\mathbf{R}^n$. To appear in Math. Nachrichten J.

[4] G. I. Arkhipov, A. A. Karacuba, and V. N. Čubarikov, Trigonometric integrals, Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 5, 971–1003, 1197.

[5] A. Carbery and J. Wright, Distributional and L^q norm inequalities for polynomials over convex bodies in $\mathbf{R}^n$, Math. Res. Lett. 8 (2001), no. 3, 233–248.

[6] R. Coifman and G. Weiss, Extension of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[7] L. Grafakos and A. Stefanov, L^p bounds for singular integrals and maximal singular integrals with rough kernel, Indiana Univ. Math. J, 47 (1998), 455–469.
[8] I. R. Parissis, Oscillatory Integrals with Polynomial Phase, Ph. D Thesis, University of Crete, 2007.
[9] M. Padimitrakis and I. R. Parissis, Singular oscillatory integrals on $\mathbf{R}^n$, Math. Z., to appear.
[10] E. M. Stein, Oscillatory integrals in Fourier analysis, Beijing lectures in harmonic analysis (Beijing, 1984), Ann. of Math. Stud., vol. 112, Princeton Univ. Press, Princeton, NJ, 1986, p. 307–355.
[11] Elias M. Stein and Stephen Wainger, The estimation of an integral arising in multiplier transformations, Studia Math. 35 (1970), 101–104.