

# Effect of Variable viscosity on Convective Heat Transfer along an Inclined Plate Embedded in Porous Medium with an Applied Magnetic Field

N.S. Tomer, Phool Singh and Manoj Kumar

**Abstract**—The flow and heat transfer characteristics for natural convection along an inclined plate in a saturated porous medium with an applied magnetic field have been studied. The fluid viscosity has been assumed to be an inverse function of temperature. Assuming temperature vary as a power function of distance. The transformed ordinary differential equations have solved by numerical integration using Runge-Kutta method. The velocity and temperature profile components on the plate are computed and discussed in detail for various values of the variable viscosity parameter, inclination angle, magnetic field parameter, and real constant ( $\lambda$ ). The results have also been interpreted with the aid of tables and graphs. The numerical values of Nusselt number have been calculated for the mentioned parameters.

**Keywords**—Heat Transfer, Magnetic Field, Porosity, Viscosity

## I. INTRODUCTION

NATURAL convection flow over vertical surfaces immersed in porous media has paramount importance because of its potential applications in soil physics, geohydrology, and filtration of solids from liquids, chemical engineering and biological systems. Study of fluid flow in porous medium is based upon the empirically determined Darcy's law. Such flows are considered to be useful in diminishing the free convection, which would otherwise occur intensely on a vertical heated surface. In addition, recent developments in modern technology have intensified more interest of many researchers in studies of heat and mass transfer in fluids due to its wide applications in geothermal and oil reservoir engineering as well as other geophysical and astrophysical studies.

Bejan and Khair [1] have studied the phenomenon of natural convection heat and mass transfer near a vertical surface embedded in fluid saturated porous medium. The effect of buoyancy force and thermal radiation in MHD boundary layer visco-elastic fluid flow over continuously moving stretching surface embedded in porous medium have been analyzed by Abel et al. [2]. An integral approach to the heat and mass transfer by natural convection from vertical plates with variable wall temperature and concentration in

porous media saturated with an electrically conducting fluid in the presence of transverse magnetic field has been studied by Cheng [3]. Sarangi et. al. [4] analyzed the unsteady free convective MHD flow and mass transfer of a viscous, incompressible, electrically conducting fluid past an infinite vertical, non-conducting porous plate with variable temperature. EL-Kabeir et. al. [5] analyzed natural convection from a permeable sphere embedded in a variable porosity porous medium due to thermal dispersion. Dual solutions in mixed convection flow near a stagnation point on a vertical porous plate are investigated by Ishak et. al. [6]. The steady two-dimensional laminar forced flow and heat transfer of a viscous incompressible electrically conducting and heat-generating fluid past a permeable wedge embedded in non-Darcy high-porosity ambient medium with uniform surface heat flux has been studied by Rashad and Bakier [7]. Abel et. al. [8] analyzed MHD flow and heat transfer to a laminar liquid film from a horizontal stretching surface. The effect of radiation on the heat and fluid flow over an unsteady stretching surface has been analyzed by El-Aziz [9]. Singh et. al. [10] studied the heat transfer over stretching surface in porous media with transverse magnetic field. Singh et. al. [11] and [12] also investigated MHD oblique stagnation-point flow towards a stretching sheet with heat transfer for steady and unsteady cases. Elbashbeshy et. al. [13] investigated the effects of thermal radiation and magnetic field on unsteady boundary layer mixed convection flow and heat transfer problem from a vertical porous stretching surface. The unsteady laminar magnetohydrodynamic (MHD) flow over a continuously stretching surface has been investigated by Ishak [14]. El-Aziz [15] investigated the flow and heat transfer over an unsteady stretching surface with Hall effect. Postelnicu [16] studied heat and mass transfer by natural convection at a stagnation point in a porous medium considering Soret and Dufour effects.

The objective of the present study is to investigate the effect of various parameters like variable viscosity parameter, inclination angle, magnetic field parameter, and real constant ( $\lambda$ ) on convective heat transfer along an inclined plate embedded in porous medium. The governing non-linear partial differential equations are first transformed into a dimensionless form and thus resulting non-similar set of equations has been solved using Shooting method technique. Results are presented graphically and discussed quantitatively for parameter values of practical interest from physical point of view.

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II. FORMULATION OF PROBLEM

An inclined flat plate embedded in porous medium with applied magnetic field of uniform strength in the direction normal to the plate is considered. It is assumed that the porous medium is homogenous and present everywhere in local thermodynamic equilibrium. The density of the medium is considered as a function of temperature and the fluid viscosity is assumed to be an inverse function of temperature. Rest of the properties of the fluid and the porous medium are assumed to be constant. The governing equations for this model, based on Boussinesq approximation, are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial p}{\partial x} + g_x \rho + \frac{\mu}{k^*} u + \sigma B_0^2 u = 0, \tag{2}$$

$$\frac{\partial p}{\partial y} + \frac{\mu}{k^*} v + \sigma B_0^2 v = 0, \tag{3}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right), \tag{4}$$

$$\rho = \rho_\infty [1 - \beta(T - T_\infty)], \tag{5}$$

$$\frac{1}{\mu} = \frac{1}{\mu_\infty} [1 + \gamma(T - T_\infty)],$$

$$\text{or } \frac{1}{\mu} = a(T - T_e) \tag{6}$$

where  $u$  and  $v$  are velocity components along  $x$  and  $y$  axes respectively,  $\mu$  is dynamic viscosity,  $\sigma$  is electrical conductivity,  $p$  is the pressure,  $T$  is temperature of the plate,  $\rho$  is density of the fluid,  $\beta$  is coefficient of thermal expansion,  $g_x$  is acceleration due to buoyancy and  $\alpha$  is the thermal diffusivity,  $B_0$  is constant magnetic field applied in  $y$ -direction,  $k^*$  is the permeability of medium,  $a = \gamma / \mu_\infty$ , and  $T_e = T_\infty - 1/\gamma$ .

Both  $a$  and  $T_e$  are constant and their values depends on the reference state and thermal property of the fluid. In general  $a > 0$  for liquids and  $a < 0$  for gases.

The wall temperature of the plate is considered as a power function of the distance from the origin. Hence,

$$\text{at } y = 0, \quad T_w = T_\infty + Ax^\lambda, \tag{7}$$

where  $A$  is a constant  $> 0$ , and  $\lambda$  is a real number.

The boundary conditions applicable to the above model are

$$\text{at } y = 0, \quad v = 0,$$

$$\text{at } y \rightarrow \infty, \quad u = 0,$$

$$\text{at } y \rightarrow \infty, \quad T = T_\infty,$$

III. METHOD OF SOLUTION

Introducing the stream function  $\Psi$  and eliminating pressure from (2) and (3)

$$-g_x \frac{\partial \rho}{\partial y} + \frac{1}{k^* a (T - T_e)^2} \frac{\partial \Psi}{\partial y} \frac{\partial T}{\partial y} = \left( \frac{1}{k^* a (T - T_e)} + \sigma B_0^2 \right) \left( \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right), \tag{8}$$

The energy equation (4), in terms of  $\Psi$  can be written as

$$\alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) = \frac{\partial T}{\partial x} \left( \frac{\partial \Psi}{\partial y} \right) - \frac{\partial T}{\partial y} \left( \frac{\partial \Psi}{\partial x} \right), \tag{9}$$

The corresponding boundary conditions are

$$\left. \begin{aligned} \text{at } y = 0, \quad \frac{\partial \Psi}{\partial x} = 0 \text{ and } T_w = T_\infty + Ax^\lambda, \\ \text{at } y \rightarrow \infty, \quad \frac{\partial \Psi}{\partial y} = 0, \\ \text{at } y \rightarrow \infty, \quad T = T_\infty, \end{aligned} \right\} \tag{10}$$

Applying boundary layer approximations invoked by earlier investigation, the equations (8) and (9) reduced to

$$-g_x \frac{\partial \rho}{\partial y} + \frac{1}{k^* a (T - T_e)^2} \frac{\partial \Psi}{\partial y} \frac{\partial T}{\partial y} = \left( \frac{1}{k^* a (T - T_e)} + \sigma B_0^2 \right) \frac{\partial^2 \Psi}{\partial y^2}, \tag{11}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha} \left( \frac{\partial T}{\partial x} \left( \frac{\partial \Psi}{\partial y} \right) - \frac{\partial T}{\partial y} \left( \frac{\partial \Psi}{\partial x} \right) \right), \tag{12}$$

The dimensionless variables introduced to obtain the similarity solutions for the equations (11) and (12) are:

$$\eta = (Ra_x)^{1/2} \left( \frac{y}{x} \right), \tag{13}$$

$$\Psi = \alpha (Ra_x)^{1/2} f(\eta), \tag{14}$$

$$\theta(\eta) = \frac{(T - T_\infty)}{(T_w - T_\infty)}, \tag{15}$$

where  $Ra_x = \rho_\infty g_x \beta k^* (T_w - T_\infty) x / \mu_\infty \alpha$  is the modified local Rayleigh number in a porous medium.

Porous parameter

$$K^2 = L^2 / k^*, \quad M^2 = B_0^2 L^2 \sigma / \mu, \quad g_x = g \cos \Omega.$$

In terms of new variable equations (10) and (11) become

$$f''(\eta) = \frac{1}{\left[ 1 - \frac{M^2}{k^2} \left( \frac{\theta - \theta_e}{\theta_e} \right) \right]} \left[ \frac{1}{(\theta - \theta_e)} f'(\eta) \theta'(\eta) + \left( \frac{\theta - \theta_e}{\theta_e} \right) \theta'(\eta) \right], \tag{16}$$

$$\theta''(\eta) + ((1 + \lambda)/2) f(\eta) \theta'(\eta) - \lambda f'(\eta) \theta(\eta) = 0, \tag{17}$$

where  $\theta_e$  is a constant is defined by

$$\theta_e = \frac{T_e - T_\infty}{T_w - T_\infty} = -\frac{1}{\gamma(T_e - T_\infty)}$$

Its value is determined by the viscosity of the fluid in the consideration and the operating temperature difference. It's also important to note that  $\theta_e$  is negative for liquids and positive for gases. The boundary conditions are

at  $\eta = 0, \quad f(\eta) = 0, \quad \text{and} \quad \theta(\eta) = 1, \quad (18)$

at  $\eta \rightarrow \infty, \quad f'(\eta) = 0, \quad \text{and} \quad \theta(\eta) = 0. \quad (19)$

The resulting differential equations can then be easily integrated, without any iteration, by initial value solver. For this purpose the well-known Runge-Kutta integration scheme has been used.

The boundary layer thickness can be obtained by defining the edge of boundary layer as the point where  $T \cong T_\infty$ , which yields

$$\frac{\delta}{x} = \frac{\eta_b}{(Ra_x)^{1/2}},$$

From Figs. 2 and 3, It is observed that the dimensionless temperature  $\theta$  has a maximum value of 1 at  $\eta = 0$  and decreases as  $\eta$  increases. Further, it is interesting to note that the rate of change of  $\theta$  increases with decreasing value of  $M$ ,  $\theta_e$  and also with increasing value of  $\lambda$ . It implies that boundary layer thickness decreases with decreasing values of  $M$ ,  $\theta_e$  and  $\lambda$ . To assess the heat transfer ability of the medium, the local Nusselt number and local heat transfer rate are defined as

$$Nu_x = \frac{hx}{k_m} = \frac{qx}{k_m(T_w - T_\infty)}, \quad (21)$$

$$q = -k_m \left( \frac{\partial T}{\partial y} \right)_{y=0}, \quad (22)$$

with the help of (6), (12) and (14), (21) can be written as

$$q = k_m A^{3/2} \left( \frac{\rho_\infty g \beta K^*}{\mu_\infty \alpha} \right)^{1/2} x^{(3\lambda - 1)/2} (-\theta'(0)), \quad (23)$$

From (21) and (23), it follows that

$$\frac{Nu_x}{(Ra_x)^{1/2}} = (-\theta'(0)). \quad (24)$$

#### IV. RESULT AND DISCUSSION

The solution of the differential equations were obtained for different values of  $M, k, \theta_e$  and  $\lambda$ . The dimensionless velocity profile and temperature profile for different values of  $\theta_e, M = 2.0, k = 1.0$  and  $\lambda = 1.0$  with respect to  $\eta$  are shown in Fig. 1, Fig. 2 respectively. The Fig. 1 indicates that  $f'$  increases as  $\theta_e \rightarrow 0$  for  $\theta_e < 0$ . The variation of viscosity with temperature has a substantial effect on heat transfer characteristics as well as the velocity and temperature distributions within the boundary layer over an inclined flat plate.

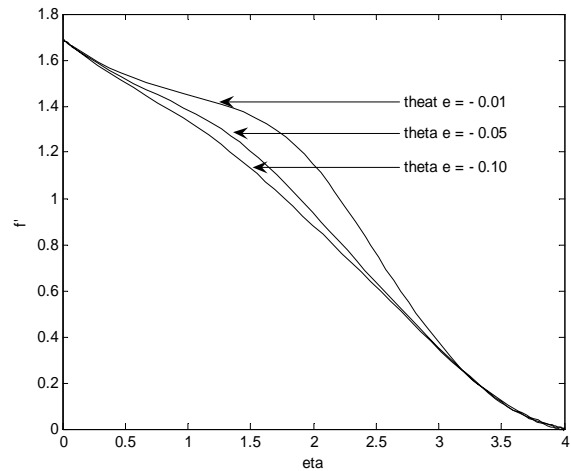


Fig. 1 Velocity profile for different values of  $\theta_e$

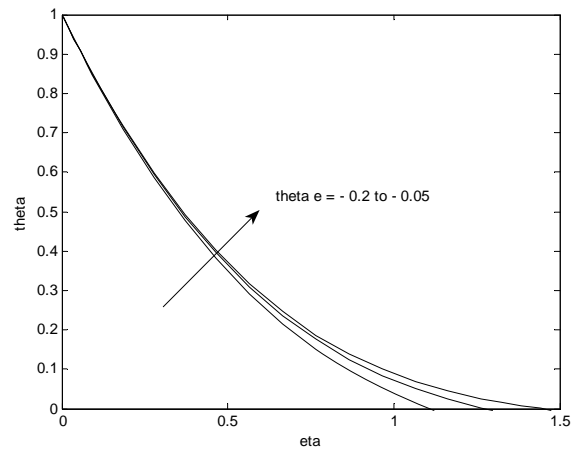


Fig. 2 Temperature profile for different values of  $\theta_e$

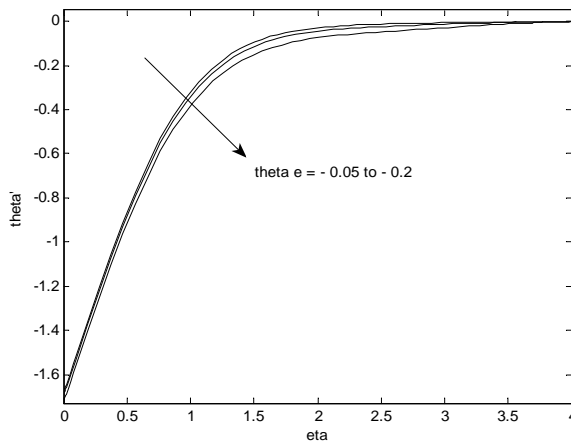


Fig. 3 Variation of Heat flux for different values of  $\theta_e$

The dimensionless temperature profile for different values of  $\theta_e$  increases with increasing value  $\theta_e$ . It has been noticed from Fig. 3, for  $\theta_e < 0$ , the local heat flux is increased considerably as  $\theta_e \rightarrow 0$ .

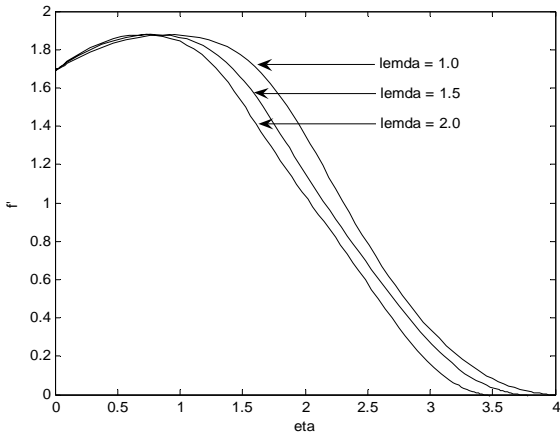


Fig. 4 Velocity profile for different values of  $\lambda$

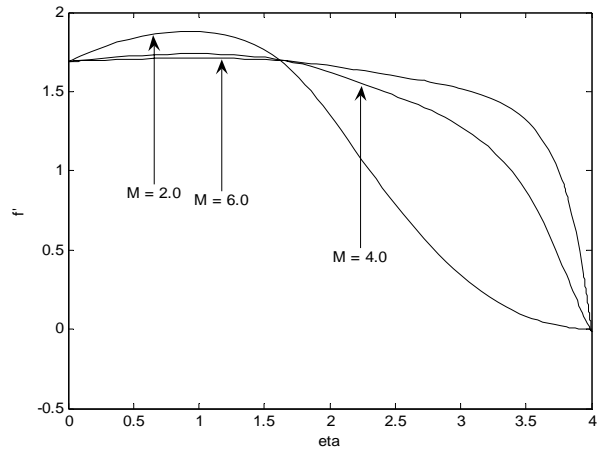


Fig. 7 Variation of velocity profile for different values of  $M$

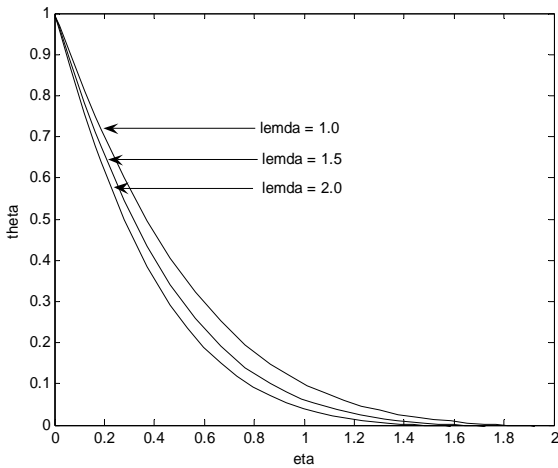


Fig. 5 Temperature profile for different values of  $\lambda$

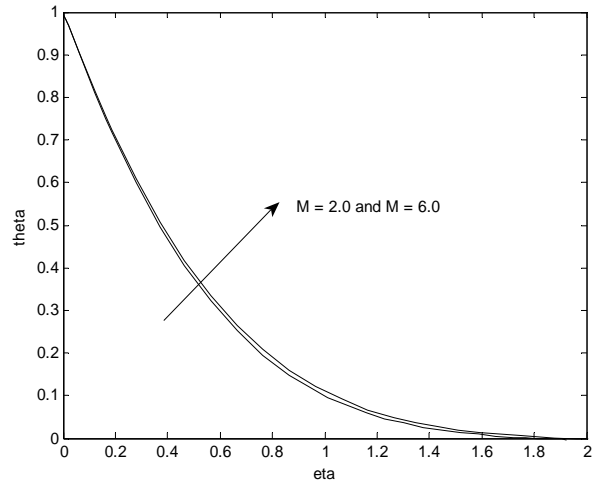


Fig. 8 Variation of temperature profile for different values of  $M$

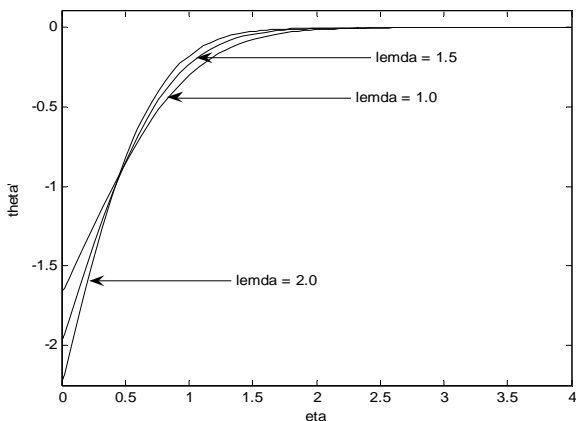


Fig. 6 Heat flux profile for different values of  $\lambda$

The dimensionless velocity profile for different values of  $\lambda$  ( $M = 2.0, k = 1.0, \theta_e = -0.01$ ) and for different values of  $M$  ( $\lambda = 1.0, k = 1.0, \theta_e = -0.01$ ) have shown in Figs. 4-8, respectively. Fig. 4 shows that the velocity profile decrease with increasing the value of  $\lambda$ . It has been observed from Figs. 5 and 6, the temperature profile is decreased with increasing value of  $\lambda$  and the heat flux profile increases with increased value of  $\lambda$ . This is suggested that in a particular porous medium, the heat transfer is governed by the intensity of the magnetic field. The boundary layer thickness is very small at lower magnetic fields and is very difficult to detect the exploratory drilling of hot water wells etc.

The velocity and temperature profile for different values of  $M$  has been shown in Figs. 7 and 8. It has been observed from the Fig. 7 that the velocity profile decreases with increased value of magnetic field parameter. It has also been noticed from Fig. 8 that the temperature profile increases with increase in the magnetic field parameter. It is observed that in particular porous medium at given location fluid temperature is high at higher magnetic field. The values of  $(-\theta'(0))$  for

different values of  $M$ ,  $k$ ,  $\theta_e$  and  $\lambda$  are presented in Table I.

TABLE I  
VALUES OF  $[-\theta'(0)]$ , FOR VARIOUS VALUES OF PARAMETERS ( $k = 1.0$ )

$M$	$\theta_e$	$\lambda$	$-\theta'(0)$
2.0	-0.01	1.0	1.67157
2.0	-0.05	1.0	1.67902
2.0	-0.10	1.0	1.69188
2.0	-0.20	1.0	1.72082
2.0	-0.01	1.5	1.97937
2.0	-0.01	2.0	2.24576
4.0	-0.01	1.0	1.64188
6.0	-0.01	1.0	1.63622

The local Nusselt number with respect to  $\lambda$  and for different values of  $M$ ,  $k$  and  $\theta_e$  has been calculated with the help of  $-\theta'(0)$  given in the Table I. It has been observed from the Table I, that with the decreasing value of  $M$  and  $\theta_e$ , the Nusselt number increases, i.e., the convective heat transfer coefficient increases. The convective rate of heat transfer also increases with increasing value of  $\lambda$ .

#### V. CONCLUSION

The flow and heat transfer characteristics for natural convection along an inclined plate in a saturated porous medium with an applied magnetic field have been studied. Fluid viscosity has been assumed to be an inverse function of temperature. Assuming temperature vary as a power function of distance. Numerical solution for the governing equations has been obtained which allows the computation of the flow and heat transfer characteristics for various values of the variable viscosity parameter, magnetic field parameter, and real constant ( $\lambda$ ). The main results of the paper can be summarized as follows:

1. The dimensionless velocity and temperature profile increases with increase in variable viscosity.
2. The dimensionless velocity and temperature profile decreases with increase in the real constant ( $\lambda$ ).
3. The dimensionless velocity decreases with increasing value of magnetic field parameter up to certain level and then increases, while the temperature profile increases with increasing value of magnetic field parameter.
4. Convective heat transfer coefficient increases with decreasing value of magnetic field parameter and variable viscosity parameter, and increases with increase in real constant ( $\lambda$ ).

These results have possible technological applications in liquid-based systems, and are expected to be very useful for practical applications.

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# Mean Codeword Lengths and Their Correspondence with Entropy Measures

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**Abstract**—The objective of the present communication is to develop new genuine exponentiated mean codeword lengths and to study deeply the problem of correspondence between well known measures of entropy and mean codeword lengths. With the help of some standard measures of entropy, we have illustrated such a correspondence. In literature, we usually come across many inequalities which are frequently used in information theory. Keeping this idea in mind, we have developed such inequalities via coding theory approach.

**Keywords**—Codeword, Code alphabet, Uniquely decipherable code, Mean codeword length, Uncertainty, Noiseless channel

## I. INTRODUCTION

IN coding theory, one of the many applications of notion of uncertainty will be to the problem of efficient coding of messages to be sent over a noiseless channel, that is, our only concern is to maximize the number of messages that can be sent over the channel in a given time. Let us assume that the messages to be transmitted are generated by a random variable  $X$  and each value  $x_i$ ,  $i = 1, 2, \dots, n$  of  $X$  must be represented by a finite sequence of symbols chosen from the set  $\{a_1, a_2, \dots, a_D\}$ . This set is called code alphabet or set of code characters and sequence assigned to each  $x_i$ ,  $i = 1, 2, \dots, n$  is called code word. Let  $n_i$  be the length of code word associated with  $x_i$  satisfying Kraft's [10] inequality given by the following mathematical expression:

$$D^{-l_1} + D^{-l_2} + \dots + D^{-l_n} \leq 1 \tag{1}$$

Where,  $D$  is the size of alphabet. In calculating the long run efficiency of communications, we choose codes to minimize average code word length, given by

$$L = \sum_{i=1}^n p_i n_i \tag{2}$$

Where,  $p_i$  is the probability of occurrence of  $x_i$ . For uniquely decipherable codes, Shannon's [13] noiseless coding theorem which states that

$$\frac{H(P)}{\log D} \leq L < \frac{H(P)}{\log D} + 1 \tag{3}$$

Determines the lower and upper bounds on  $L$  in terms of Shannon's [13] entropy  $H(P)$ . Campbell [5] for the first

time introduced the idea of exponentiated mean codeword length for uniquely decipherable codes and proved a noiseless coding theorem. He considered a special exponentiated mean of order  $\alpha$  given by

$$L_\alpha = \frac{\alpha}{1-\alpha} \log_D \left[ \sum_{i=1}^n p_i D^{(1-\alpha)n_i/\alpha} \right] \tag{4}$$

and showed that its lower bound lies between  $R_\alpha(P)$  and

$R_\alpha(P) + 1$  where  $R_\alpha(P)$  is expressed as:

$$R_\alpha(P) = (1-\alpha)^{-1} \log_D \left[ \sum_{i=1}^n p_i^\alpha \right]; \alpha > 0, \alpha \neq 1 \tag{5}$$

The above is Renyi's [12] measure of entropy of order  $\alpha$ . As  $\alpha \rightarrow 1$ , it is easily shown that  $L_\alpha \rightarrow L$  and  $R_\alpha(P)$  approaches  $H(P)$ .

Guiasu and Picard [6] defined the weighted average length for a uniquely decipherable code as

$$\bar{L} = \sum_{i=1}^n \left( \frac{u_i n_i p_i}{\sum_{i=1}^n u_i p_i} \right) \tag{6}$$

Longo [11] interpreted (6) as the average cost of transmitting letters  $x_i$  with probability  $p_i$  and utility  $u_i$  and provided some practical interpretation of this length and also derived the lower and upper bounds for the cost function (6).

In the literature of coding theory, for the given value of an inequality, say, Kraft's [10] inequality, we have a pair of problems. For a given mean codeword length, we can find its lower bounds for all uniquely decipherable codes. In the inverse problem, we can find the mean value for the given pair of lower bounds. The direct problem has a unique answer, though it may not always be easy to find an analytical expression for it. However, the inverse problem has no unique answer in the sense that the same lower bounds may arise for a number of means. The challenge is to find as many of the means as possible, which have the given pair of values as lower bounds and this is the theme of the present communication.

In section II, we have developed two new mean codeword lengths with the help of divergence measures. In section III, we have illustrated the correspondence between standard measures of entropy and the codeword lengths. The development of new inequalities has been made in section IV.

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II. DEVELOPMENT OF TWO NEW MEAN CODEWORDS

A. For the development of first mean, we consider Sharma and Mittal's [15] measure of directed divergence given by

$$D_r^s(P \square Q) = \frac{1}{s-1} \left[ \left( \sum_{i=1}^n p_i^r q_i^{1-r} \right)^{\frac{s-1}{r-1}} - 1 \right] \geq 0 \quad (7)$$

Putting  $q_i = \frac{D^{-l_i}}{\sum_{i=1}^n D^{-l_i}}$  in equation (7), we get

$$\left( \sum_{i=1}^n p_i^r D^{-l_i(1-r)} \right)^{\frac{s-1}{r-1}} \geq \left( \sum_{i=1}^n D^{-l_i} \right)^{-(s-1)} \quad (8)$$

Taking logarithms and then dividing both sides of equation (8) by  $(s-1)$ , we get the following expression:

$$\begin{aligned} & \frac{1}{r-1} \log_D \left( \frac{\sum_{i=1}^n p_i^r D^{l_i(r-1)}}{\sum_{i=1}^n p_i^r} \right) + \log_D \sum_{i=1}^n D^{-l_i} \\ & \geq \frac{1}{1-r} \log_D \left( \sum_{i=1}^n p_i^r \right) \end{aligned}$$

that is

$$\bar{L}_r + \log_D \sum_{i=1}^n D^{-l_i} \geq R_r(P)$$

where

$$\bar{L}_r = \frac{1}{r-1} \log_D \left( \frac{\sum_{i=1}^n p_i^r D^{l_i(r-1)}}{\sum_{i=1}^n p_i^r} \right) \quad (9)$$

Since  $\sum_{i=1}^n D^{-l_i}$  lies between  $D^{-1}$  and 1, so lower bound of

$\bar{L}_r$  lies between  $R_r(P)$  and  $R_r(P)+1$  where  $R_r(P)$  is a Renyi's [12] measure of entropy of order  $r$ .

It can easily be proved that: (i) When  $l_1 = l_2 = \dots = l_n = l$ ,

then  $\bar{L}_r = l$

(ii)  $\bar{L}_r$  lies between minimum and maximum values of  $l_1, l_2, \dots, l_n$

(iii) When  $r \rightarrow 1$ , then  $\bar{L}_r \rightarrow L$  where  $L = \sum_{i=1}^n p_i l_i$  (9)

Thus, the mean codeword length introduced in equation is a genuine mean codeword length as it satisfies the essential

properties of being a mean codeword length.

B. For the development of second mean, we consider Bhattacharya's [2] measure of directed divergence is given by

$$D(P, Q) = \sum_{i=1}^n (\sqrt{p_i} - \sqrt{q_i})^2$$

or

$$D(P, Q) = 1 - \sum_{i=1}^n \sqrt{p_i} \sqrt{q_i} \geq 0 \quad (10)$$

Putting  $q_i = \frac{D^{-l_i}}{\sum_{i=1}^n D^{-l_i}}$  in equation (10), we get

$$\left( \sum_{i=1}^n D^{-l_i} \right)^{\frac{1}{2}} \geq \sum_{i=1}^n \sqrt{p_i} (D^{-l_i})^{\frac{1}{2}} \quad (11)$$

Taking Logarithms on both sides of (11), we get

$$\begin{aligned} & \log_D \left( \sum_{i=1}^n D^{-l_i} \right) \geq \\ & 2 \log_D \left( \frac{\sum_{i=1}^n \sqrt{p_i} (D^{-l_i})^{\frac{1}{2}}}{\sum_{i=1}^n \sqrt{p_i}} \right) + 2 \log_D \left( \sum_{i=1}^n \sqrt{p_i} \right) \end{aligned}$$

or

$$\begin{aligned} & \frac{1}{\frac{1}{2}-1} \log_D \left( \frac{\sum_{i=1}^n p_i^{\frac{1}{2}} D^{l_i(\frac{1}{2}-1)}}{\sum_{i=1}^n p_i^{\frac{1}{2}}} \right) + \log_D \left( \sum_{i=1}^n D^{-l_i} \right) \\ & \geq \frac{1}{1-\frac{1}{2}} \log_D \left( \sum_{i=1}^n p_i^{\frac{1}{2}} \right) \end{aligned}$$

that is

$$\bar{L}_{\frac{1}{2}} \geq R_{\frac{1}{2}}(P) - \log_D \left( \sum_{i=1}^n D^{-l_i} \right)$$

where

$$\bar{L}_{\frac{1}{2}} = \frac{1}{\frac{1}{2}-1} \log_D \left( \frac{\sum_{i=1}^n p_i^{\frac{1}{2}} D^{l_i(\frac{1}{2}-1)}}{\sum_{i=1}^n p_i^{\frac{1}{2}}} \right) \quad (12)$$

Since  $\sum_{i=1}^n D^{-l_i}$  lies between  $D^{-1}$  and 1, so lower bound of

$\bar{L}_{\frac{1}{2}}$  lies between  $R_{\frac{1}{2}}(P)$  and  $R_{\frac{1}{2}}(P)+1$  where  $R_{\frac{1}{2}}(P)$  is

a Renyi's [12] measure of entropy of order  $\frac{1}{2}$ .

Thus, the mean codeword length introduced in equation (12) is a genuine mean codeword length as it can easily be proved that it satisfies the essential properties of being a mean codeword length.

### III. CORRESPONDENCE BETWEEN MEAN CODEWORD LENGTHS AND THE ENTROPY MEASURES

The object of the present paper is to go deeper into the problem of correspondence between well known measures of entropy and mean codeword lengths. We state the results in a broader framework as follows:

(a) To every mean codeword length, there corresponds a measure of entropy or a monotonic increasing function of a measure of entropy.

(b) To every measure of entropy, there corresponds a mean codeword length or a monotonic increasing function of the mean codeword length.

For many purposes, especially for maximization of entropy purposes, every monotonic increasing function of a measure of entropy is as good as a measure of entropy and for such purposes; all such functions should be regarded as equivalent. A monotonic increasing function of mean codeword lengths is not the same as a mean codeword length, but minimizing a monotonic increasing function of a mean codeword length gives the same results as minimizing the mean codeword length itself. Thus, we do not lose anything significant from our results by using monotonic increasing functions of entropy and mean codeword lengths. Below, we illustrate the correspondence between standard measures of entropy and the codeword lengths:

*Theorem:* For all uniquely decipherable codes, the lower bound of exponentiated mean codeword length

$$L(\alpha, \beta) = \frac{\beta - \alpha + 1}{\alpha - \beta} \log_D \left( \sum_{i=1}^n p_i D^{l_i \left( \frac{\beta - \alpha + 1}{\alpha - \beta} \right)} \right)$$

lies between

$$\frac{1}{\alpha - \beta} \log_D \left( \sum_{i=1}^n p_i^{\beta - \alpha + 1} \right) \text{ and } \frac{1}{\alpha - \beta} \log_D \left( \sum_{i=1}^n p_i^{\beta - \alpha + 1} \right) + 1,$$

$$\alpha - 1 < \beta < \alpha, \alpha \geq 1$$

where

$$\frac{1}{\alpha - \beta} \log_D \left( \sum_{i=1}^n p_i^{\beta - \alpha + 1} \right)$$

is the Varma's [16] measure of entropy.

*Proof:* Here we use Holder's inequality

$$\sum_{i=1}^n x_i y_i \geq \left( \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n y_i^q \right)^{\frac{1}{q}}, \frac{1}{p} + \frac{1}{q} = 1,$$

$$p \text{ or } q < 1 \tag{13}$$

Substituting

$$x_i = p_i^{-\left( \frac{\beta - \alpha + 1}{\alpha - \beta} \right)} D^{-l_i}, y_i = p_i^{\left( \frac{\beta - \alpha + 1}{\alpha - \beta} \right)}$$

$$\frac{1}{p} = \frac{\alpha - \beta - 1}{\alpha - \beta},$$

$$\frac{1}{q} = \frac{1}{\alpha - \beta} \text{ so that}$$

$$\frac{1}{p} + \frac{1}{q} = 1 \text{ and either } p \text{ or } q < 1 \text{ in equation (13), we get}$$

$$\sum_{i=1}^n D^{-l_i} \geq \left[ \sum_{i=1}^n \left( p_i D^{-l_i \left( \frac{\alpha - \beta}{\alpha - \beta - 1} \right)} \right) \right]^{\frac{\alpha - \beta - 1}{\alpha - \beta}} \left[ \sum_{i=1}^n p_i^{(\beta - \alpha + 1)} \right]^{\frac{1}{\alpha - \beta}} \tag{14}$$

Taking log on both sides of (14), we get

$$\left( \frac{\alpha - \beta - 1}{\beta - \alpha} \right) \log_D \sum_{i=1}^n \left( p_i D^{l_i \left( \frac{\beta - \alpha}{\alpha - \beta - 1} \right)} \right) \geq \frac{1}{\alpha - \beta} \log_D \sum_{i=1}^n p_i^{(\beta - \alpha + 1)} \tag{15}$$

So that, if  $l_i$ 's are to be integers, the lower bound for

$$L(\alpha, \beta) \text{ lies between } \frac{1}{\alpha - \beta} \log_D \sum_{i=1}^n p_i^{(\beta - \alpha + 1)} \text{ and}$$

$$\frac{1}{\alpha - \beta} \log_D \sum_{i=1}^n p_i^{(\beta - \alpha + 1)} + 1$$

The R.H.S of (15) is Varma's [16] measure of entropy and L.H.S is a genuine mean codeword length as it satisfies the essential properties of being a mean codeword length.

It has been proved that Shannon's [13] entropy, Renyi's [12] entropy of order , Kapur's [8] entropy of order and type all provide lower bounds for different mean codeword lengths, while Havrda and Charvat's [7], Arimoto's [1] and Behara and Chawla's [3] measures of entropy provide lower bounds for some monotonic increasing functions of mean codeword lengths but not for mean codeword length's themselves. Below, we discuss the correspondence between standard measures of entropy and the possible lower bounds:

#### A. Kapur's [8] measure of entropy as a possible lower bound

Kapur's [8] measure of entropy of order  $\alpha$  and type  $\beta$  is given by

$$K_{\alpha}^{\beta}(P) = \frac{1}{\alpha + \beta - 2} \left( \sum_{i=1}^n p_i^{\alpha} + \sum_{i=1}^n p_i^{\beta} - 2 \right) \tag{16}$$

We know that

$$\frac{1}{1-\alpha} \log_D \sum_{i=1}^n p_i^\alpha \leq \frac{1}{\alpha-1} \log_D \left( \frac{\sum_{i=1}^n p_i^\alpha D^{(\alpha-1)l_i}}{\sum_{i=1}^n p_i^\alpha} \right)$$

Where L.H.S is Renyi's [12] entropy of order  $\alpha$  and R.H.S is an exponentiated mean of order  $\alpha$  developed in equation (9). that is,

$$\log_D \sum_{i=1}^n p_i^\alpha \leq -\log_D \left( \frac{\sum_{i=1}^n p_i^\alpha D^{(\alpha-1)l_i}}{\sum_{i=1}^n p_i^\alpha} \right)$$

which gives

$$\sum_{i=1}^n p_i^\alpha \leq \frac{\sum_{i=1}^n p_i^\alpha}{\sum_{i=1}^n p_i^\alpha D^{(\alpha-1)l_i}}$$

Similarly,

$$\sum_{i=1}^n p_i^\beta \leq \frac{\sum_{i=1}^n p_i^\beta}{\sum_{i=1}^n p_i^\beta D^{(\beta-1)l_i}}$$

Equality sign holds in both cases when  $D^{-l_i} = p_i$ . Thus, we have

$$\left( \sum_{i=1}^n p_i^\alpha + \sum_{i=1}^n p_i^\beta - 2 \right) \leq \left( \frac{\sum_{i=1}^n p_i^\alpha}{\sum_{i=1}^n p_i^\alpha D^{(\alpha-1)l_i}} + \frac{\sum_{i=1}^n p_i^\beta}{\sum_{i=1}^n p_i^\beta D^{(\beta-1)l_i}} - 2 \right)$$

Dividing both sides by  $\alpha + \beta - 2$ , we get

$$\frac{1}{\alpha + \beta - 2} \left( \sum_{i=1}^n p_i^\alpha + \sum_{i=1}^n p_i^\beta - 2 \right) \leq \frac{1}{\alpha + \beta - 2} \left( \frac{\sum_{i=1}^n p_i^\alpha}{\sum_{i=1}^n p_i^\alpha D^{(\alpha-1)l_i}} + \frac{\sum_{i=1}^n p_i^\beta}{\sum_{i=1}^n p_i^\beta D^{(\beta-1)l_i}} - 2 \right) \quad (17)$$

The L.H.S. of equation (17) is Kapur's [8] measure of entropy of order  $\alpha$  and type  $\beta$  but R.H.S. is neither a mean codeword length nor a monotonic increasing unction of mean codeword length.

*B. Sharma and Mittal's [14] measure of entropy as a possible lower bound*

Sharma and Mittal's [14] measure of entropy is given by

$$\phi(P) = \frac{1}{2^{1-\alpha} - 1} \left[ \left( \sum_{i=1}^n p_i^\beta \right)^{\frac{\alpha-1}{\beta-1}} - 1 \right], \alpha \neq 1, \beta \neq 1$$

$$\alpha > 0, \beta > 0 \quad (18)$$

From equation (9), we have

$$\frac{1}{1-\beta} \log_D \sum_{i=1}^n p_i^\beta \leq \frac{1}{\beta-1} \log_D \left( \frac{\sum_{i=1}^n p_i^\beta D^{(\beta-1)l_i}}{\sum_{i=1}^n p_i^\beta} \right)$$

If  $\beta < 1$ , we have

$$\log_D \sum_{i=1}^n p_i^\beta \leq -\log_D \left( \frac{\sum_{i=1}^n p_i^\beta D^{(\beta-1)l_i}}{\sum_{i=1}^n p_i^\beta} \right)$$

which further gives

$$\Rightarrow \left[ \left( \sum_{i=1}^n p_i^\beta \right)^{\frac{\alpha-1}{\beta-1}} - 1 \right] \leq \left[ \left( \frac{\sum_{i=1}^n p_i^\beta}{\sum_{i=1}^n p_i^\beta D^{(\beta-1)l_i}} \right)^{\frac{\alpha-1}{\beta-1}} - 1 \right]$$

Dividing both sides by  $2^{1-\alpha} - 1$ , we get

$$\frac{1}{2^{1-\alpha} - 1} \left[ \left( \sum_{i=1}^n p_i^\beta \right)^{\frac{\alpha-1}{\beta-1}} - 1 \right] \leq \frac{1}{2^{1-\alpha} - 1} \left[ \left( \frac{\sum_{i=1}^n p_i^\beta}{\sum_{i=1}^n p_i^\beta D^{(\beta-1)l_i}} \right)^{\frac{\alpha-1}{\beta-1}} - 1 \right] \quad (19)$$

The L.H.S. of equation (19) is Sharma and Mittal's [14] measure of entropy but R.H.S is neither a mean codeword length nor a monotonic increasing unction of mean codeword length.

In the literature of information theory, we usually come across many inequalities which are frequently applicable and used for the manipulation of mathematical results. The following section deals with development of such inequalities and relations between measures of entropy via coding theory approach.

#### IV. NEW INEQUALITIES VIA CODING THEORY APPROACH

In this section, we generate the following inequalities by using well known measures of directed divergence:

A. Burg's [4] measure of entropy is given by

$$\varphi_1(p) = \sum_{i=1}^n \log_D p_i \quad (20)$$

Also, Burg's [4] measure of directed divergence is given by

$$D(P, Q) = \sum_{i=1}^n \left( \frac{p_i}{q_i} - \log_D \frac{p_i}{q_i} - 1 \right) \geq 0 \quad (21)$$

Putting  $q_i = \frac{D^{-l_i}}{\sum_{i=1}^n D^{-l_i}}$  in equation (21), we get

$$\sum_{i=1}^n \left( p_i \left( \frac{\sum_{i=1}^n D^{-l_i}}{D^{-l_i}} \right) - \log_D p_i \left( \frac{\sum_{i=1}^n D^{-l_i}}{D^{-l_i}} \right) - 1 \right) \geq 0$$

or

$$\sum_{i=1}^n \left( \frac{p_i}{D^{-l_i}} \sum_{i=1}^n D^{-l_i} \right) - \sum_{i=1}^n \log_D p_i - \sum_{i=1}^n \log_D \left( \frac{\sum_{i=1}^n D^{-l_i}}{D^{-l_i}} \right) - n \geq 0 \quad (22)$$

If we take  $l_1 = l_2 = \dots = l_n = l$  in (22), we get

$$\log_D \left( \frac{1}{n^n} \right) \geq \sum_{i=1}^n \log_D p_i$$

Thus, we have

$$\varphi_1(p) \leq \log_D \left( \frac{1}{n^n} \right)$$

which is a new inequality.

B. Jensen's divergence is given by

$$J(P \square Q) = \sum_{i=1}^n (p_i - q_i) \log_D \frac{p_i}{q_i}$$

We know that

$$J(P \square Q) = \sum_{i=1}^n (p_i - q_i) \log_D \frac{p_i}{q_i} \geq 0 \quad (23)$$

Putting  $q_i = \frac{D^{-l_i}}{\sum_{i=1}^n D^{-l_i}}$  in equation (23), we get

$$\sum_{i=1}^n \left( p_i - \frac{D^{-l_i}}{\sum_{i=1}^n D^{-l_i}} \right) \left( \log_D p_i + \log_D \frac{\sum_{i=1}^n D^{-l_i}}{D^{-l_i}} \right) \geq 0$$

or

$$\sum_{i=1}^n p_i \log_D p_i + \sum_{i=1}^n p_i \log_D \left( \frac{\sum_{i=1}^n D^{-l_i}}{D^{-l_i}} \right) - \sum_{i=1}^n \left( \frac{D^{-l_i}}{\sum_{i=1}^n D^{-l_i}} \right) \log_D p_i - \sum_{i=1}^n \left( \frac{D^{-l_i}}{\sum_{i=1}^n D^{-l_i}} \right) \log_D \left( \frac{\sum_{i=1}^n D^{-l_i}}{D^{-l_i}} \right) \geq 0 \quad (24)$$

If we take  $l_1 = l_2 = \dots = l_n = l$  in (24), we get

$$\log_D n - \frac{1}{n} \sum_{i=1}^n \log_D p_i - \frac{1}{n} \sum_{i=1}^n \log_D n \geq - \sum_{i=1}^n p_i \log_D p_i$$

or

$$- \sum_{i=1}^n p_i \log_D p_i + \frac{1}{n} \sum_{i=1}^n \log_D p_i \leq 0$$

Thus, we have

$$nH(P) + \varphi_1(p) \leq 0$$

which gives the relation between Shannon's [13] measure of entropy and Burg's [4] entropy.

C. Kapur's [9] measure of directed divergence is given by

$$D_\alpha(P, Q) = \frac{1}{1-\alpha} \log_D \left( \frac{\left( \sum_{i=1}^n p_i q_i^{\alpha-1} \right)^\alpha}{\left( \sum_{i=1}^n q_i^\alpha \right)^{\alpha-1} \sum_{i=1}^n p_i^\alpha} \right) \geq 0 \quad (25)$$

Putting  $q_i = \frac{D^{-l_i}}{\sum_{i=1}^n D^{-l_i}}$  in equation (25), we get

or

$$\frac{\alpha}{1-\alpha} \log_D \sum_{i=1}^n p_i D^{l_i(1-\alpha)} + \log_D \sum_{i=1}^n D^{-l_i \alpha} \geq \frac{1}{1-\alpha} \log_D \sum_{i=1}^n p_i^\alpha \quad (26)$$

If we take  $l_1 = l_2 = \dots = l_n = l$  in (26), we get

$$\frac{\alpha}{1-\alpha} \log_D D^{l(1-\alpha)} + \log_D D^{-l\alpha} n$$

$$\geq \frac{1}{1-\alpha} \log_D \sum_{i=1}^n p_i^\alpha$$

or

$$\frac{1}{1-\alpha} \log_D \sum_{i=1}^n p_i^\alpha \leq \log_D n$$

$$\text{or } R_r(P) \leq \log_D n,$$

This is a well known inequality, already existing in the literature of information theory and has been proved by some other technique.

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