Nonlinear Waves In Fluid-Filled Stenosed Elastic Tube : Nonlinear Schrödinger Equation With Variable Coefficient

Choy Yaan Yee, Tay Kim Gaik, and Ong Chee Tiong

Abstract—In the present work, by utilizing the artery as a prestressed thin-walled elastic tube with a symmetrical stenosis and the blood as an incompressible full inviscid fluid, we have studied the amplitude modulation of nonlinear waves in such a composite medium by using the reductive perturbation method. The governing equations can be reduced to nonlinear Schrödinger (NLS) equation with variable coefficient. The wave solution propagates to the left by preserving their bell-shape form while time ξ increases.

Keywords—Nonlinear waves, full inviscid fluid, elastic tube with stenosis.

I. Introduction

The propagation of pressure pulses in fluid-filled distensible tubes has been studied by several researchers due to its application in arterial mechanics [1]-[2]. The modulation of small-but-finite amplitude pressure waves in a fluid-filled distensible, linear elastic tube has been examined by Ravindran and Prasad [3]. They obtained the non-linear Schrodinger (NLS) equation. The work of of non-linear waves modulation in a prestressed thin elastic tube filled with inviscid or viscous fluid has been carried out by Demiray and co-worker [4]-[6]. They showed that the governing equations can be reduced to NLS and dissipative NLS equations, respectively. The NLS equation is the simplest representative equation describing the self-modulation of one-dimensional monochromatic plane waves in dispersive media. It has a balance between the nonlinearity and dispersion.

Recently, Tay and co-workers [7]-[9] studied the non-linear waves propagation in a prestressed thin elastic tube with a symmetrical stenonis filled with inviscid, viscous and Newtonian fluid with variable viscosity, they showed that the governing equations can be reduced to forced Kortewed-de Vries, forced perturbed Kortewed-de Vries and forced Kortewed-de Vries-Burgers equations, respectively.

In the present work, by considering the artery as an incompressible, prestressed, thin-walled elastic tube with a symmetrical stenosis and the blood as an full inviscid fluid, we showed that the amplitude modulation of pressure waves can be described by the nonlinear Schrödinger equation with variable coefficient. We then sought the progressive wave solution to the nonlinear evolution equation obtained.

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II. The Governing Equations

In this section, we shall derive the governing equations of an elastic tube filled with a inviscid fluid. Such a combination of a solid and a fluid is considered to be a model for blood flow in arteries [7]. The following non-dimensional equations of tube and fluid are shown as below

$$\begin{aligned} p \;=\; & -\frac{1}{(\lambda_\theta - f(z) + u)}\frac{\partial}{\partial z}\left\{\frac{(-f' + \frac{\partial u}{\partial z})}{[1 + (-f' + \frac{\partial u}{\partial z})^2]^{1/2}}\frac{\partial \Sigma}{\partial \lambda_1}\right\} \\ & + \frac{m}{\lambda_z(\lambda_\theta - f(z) + u)}\frac{\partial^2 u}{\partial t^2} \\ & + \frac{1}{\lambda_z(\lambda_\theta - f(z) + u)}\frac{\partial \Sigma}{\partial \lambda_2}, \end{aligned} \tag{1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + q\frac{\partial v}{\partial z} + \frac{\partial \bar{p}}{\partial r} = 0, \tag{2}$$

$$\frac{\partial q}{\partial t} + v\frac{\partial q}{\partial r} + q\frac{\partial q}{\partial z} + \frac{\partial \bar{p}}{\partial z} = 0, \tag{3}$$

$$\frac{\partial v}{\partial r} + \frac{v}{r} + \frac{\partial q}{\partial z} = 0, \tag{4}$$

with the boundary conditions

$$\begin{aligned} & v|_{r=\lambda_\theta - f(z)+u} = \frac{\partial u}{\partial t} + \left(-f' + \frac{\partial u}{\partial z}\right) q|_{r=\lambda_\theta - f(z)+u}, \\ & \bar{p}|_{r=\lambda_\theta - f(z)+u} = p. \end{aligned} \tag{5}$$

The equations (1)-(5) give sufficient relations to determine the field quantities $u, v, q,$ and p completely.

III. Nonlinear Wave Modulation

In this section, we will examine the amplitude modulation of weakly non-linear waves in a fluid-filled thin elastic tube with a stenosis whose non-dimensional governing equations are given in equations (1)-(5). Considering the dispersion relation of the linearized field equations and the nature of the problem of concern, which is a boundary-value problem, the following stretched coordinates is introduced:

$$\xi = \epsilon(z - \lambda t), \quad \tau = \epsilon^2 z, \tag{6}$$

where ϵ is a small parameter measuring the weakness of nonlinearity and λ is a constant to be determined from the solution. Assuming that the field variables $u, v, q, \bar{p}$ and p

are functions of the slow variables (ξ, τ) as well as the fast variables (z, t), the following relations hold:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} - \epsilon\lambda\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z} + \epsilon\frac{\partial}{\partial \xi} + \epsilon^2\frac{\partial}{\partial \tau}. \tag{7}$$

We will further assume that the field variables may be expanded into asymptotic series of ϵ as

$$\begin{aligned} u &= \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + ..., \\ v &= \epsilon v_1 + \epsilon^2 v_2 + \epsilon^3 v_3 + ..., \\ q &= \epsilon q_1 + \epsilon^2 q_2 + \epsilon^3 q_3 + ..., \\ \bar{p} &= \bar{p_0} + \epsilon\bar{p_1} + \epsilon^2\bar{p_2} + \epsilon^3\bar{p_3} + ..., \\ p &= p_0 + \epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + ..., \\ h(\tau) &= \epsilon^2 h_1(\tau) + \epsilon^3 h_2(\tau) + ... \end{aligned} \tag{8}$$

Introducing (7) and (8) into the equations (1)-(5), the following sets of differential equations are obtained

$O(\epsilon)$ equations

$$\begin{aligned} & p_1 = \frac{m}{\lambda_\theta\lambda_z}\frac{\partial^2 u_1}{\partial t^2} - \alpha_0\frac{\partial^2 u_1}{\partial z^2} + \beta_1 u_1, \\ & \frac{\partial v_1}{\partial t} + \frac{\partial \bar{p_1}}{\partial r} = 0, \quad \frac{\partial q_1}{\partial t} + \frac{\partial \bar{p_1}}{\partial z} = 0, \\ & \frac{\partial v_1}{\partial r} + \frac{v_1}{r} + \frac{\partial q_1}{\partial z} = 0, \end{aligned} \tag{9}$$

and the boundary conditions

$$v_1|_{r=\lambda_\theta} = \frac{\partial u_1}{\partial t}, \quad \bar{p_1}|_{r=\lambda_\theta} = p_1. \tag{10}$$

$O(\epsilon^2)$ equations

$$\begin{aligned} & p_2 = \frac{m}{\lambda_\theta\lambda_z}\frac{\partial^2 u_2}{\partial t^2} - \alpha_0\frac{\partial^2 u_2}{\partial z^2} + \beta_1(u_2 - h_1) \\ & -\frac{2m\lambda}{\lambda_\theta\lambda_z}\frac{\partial^2 u_1}{\partial \xi\partial t} - 2\alpha_0\frac{\partial^2 u_1}{\partial \xi\partial z} - \frac{m}{\lambda_\theta^2\lambda_z}u_1\frac{\partial^2 u_1}{\partial t^2} \\ & -\alpha_1\left(\frac{\partial u_1}{\partial z}\right)^2 - \left(2\alpha_1 - \frac{\alpha_0}{\lambda_\theta}\right)u_1\frac{\partial^2 u_1}{\partial z^2} + \beta_2 u_1^2, \\ & \frac{\partial v_2}{\partial t} - \lambda\frac{\partial v_1}{\partial \xi} + v_1\frac{\partial v_1}{\partial r} + q_1\frac{\partial v_1}{\partial z} + \frac{\partial \bar{p_2}}{\partial r} = 0, \\ & \frac{\partial q_2}{\partial t} - \lambda\frac{\partial q_1}{\partial \xi} + v_1\frac{\partial q_1}{\partial r} + q_1\frac{\partial q_1}{\partial z} + \frac{\partial \bar{p_2}}{\partial z} + \frac{\partial \bar{p_1}}{\partial \xi} = 0, \\ & \frac{\partial v_2}{\partial r} + \frac{v_2}{r} + \frac{\partial q_2}{\partial z} + \frac{\partial q_1}{\partial \xi} = 0, \end{aligned} \tag{11}$$

and the boundary conditions

$$\begin{aligned} & \left[u_1\frac{\partial v_1}{\partial r} + v_2\right]|_{r=\lambda_\theta} = \frac{\partial u_2}{\partial t} - \lambda\frac{\partial u_1}{\partial \xi} + \frac{\partial u_1}{\partial z}q_1|_{r=\lambda_\theta}, \\ & \left[u_1\frac{\partial \bar{p_1}}{\partial r} + \bar{p_2}\right]|_{r=\lambda_\theta} = p_2. \end{aligned} \tag{12}$$

$O(\epsilon^3)$ equations

$$\begin{aligned} & p_3 = \frac{m}{\lambda_\theta\lambda_z}\frac{\partial^2 u_3}{\partial t^2} - \alpha_0\frac{\partial^2 u_3}{\partial z^2} - \frac{2m\lambda}{\lambda_\theta\lambda_z}\frac{\partial^2 u_2}{\partial \xi\partial t} \\ & -2\alpha_0\frac{\partial^2 u_2}{\partial \xi\partial z} - \alpha_0\left(\frac{\partial^2 u_1}{\partial \xi^2} + 2\frac{\partial^2 u_1}{\partial z\partial \tau}\right) + \frac{m\lambda^2}{\lambda_\theta\lambda_z}\frac{\partial^2 u_1}{\partial \xi^2} \\ & +\beta_1(u_3 - h_2) - \frac{m}{\lambda_\theta^2\lambda_z}u_1\left(\frac{\partial^2 u_2}{\partial t^2} - 2\lambda\frac{\partial^2 u_1}{\partial \xi\partial t}\right) \\ & -\frac{m}{\lambda_\theta^2\lambda_z}(u_2 - h_1)\frac{\partial^2 u_1}{\partial t^2} - 2\alpha_1\frac{\partial u_1}{\partial z}\left(\frac{\partial u_2}{\partial z} + \frac{\partial u_1}{\partial \xi}\right) \\ & -\left(2\alpha_1 - \frac{\alpha_0}{\lambda_\theta}\right)(u_2 - h_1)\frac{\partial^2 u_1}{\partial z^2} + 2\beta_2 u_1(u_2 - h_1) \\ & -\left(2\alpha_1 - \frac{\alpha_0}{\lambda_\theta}\right)u_1\left(\frac{\partial^2 u_2}{\partial z^2} + 2\frac{\partial^2 u_1}{\partial z\partial \xi}\right) \\ & +\frac{m}{\lambda_\theta^3\lambda_z}u_1^2\frac{\partial^2 u_1}{\partial t^2} - \left(\alpha_2 - \frac{\alpha_1}{\lambda_\theta}\right)u_1\left(\frac{\partial u_1}{\partial z}\right)^2 \\ & -\left(\alpha_2 - \frac{2\alpha_1}{\lambda_\theta} + \frac{\alpha_0}{\lambda_\theta^2}\right)u_1^2\frac{\partial^2 u_1}{\partial z^2} + \beta_3 u_1^3 \\ & -3\left(\gamma_1 - \frac{\alpha_0}{2}\right)\left(\frac{\partial u_1}{\partial z}\right)^2\frac{\partial^2 u_1}{\partial z^2}, \\ & \frac{\partial v_3}{\partial t} - \lambda\frac{\partial v_2}{\partial \xi} + v_1\frac{\partial v_2}{\partial r} + v_2\frac{\partial v_1}{\partial r} + q_1\frac{\partial v_2}{\partial z} + q_1\frac{\partial v_1}{\partial \xi} \\ & +q_2\frac{\partial v_1}{\partial z} + \frac{\partial \bar{p_3}}{\partial r} = 0, \\ & \frac{\partial q_3}{\partial t} - \lambda\frac{\partial q_2}{\partial \xi} + v_1\frac{\partial q_2}{\partial r} + v_2\frac{\partial q_1}{\partial r} + q_1\frac{\partial q_2}{\partial z} + q_1\frac{\partial q_1}{\partial \xi} \\ & +q_2\frac{\partial q_1}{\partial z} + \frac{\partial \bar{p_3}}{\partial z} + \frac{\partial \bar{p_2}}{\partial \xi} + \frac{\partial \bar{p_1}}{\partial \tau} = 0, \\ & \frac{\partial v_3}{\partial r} + \frac{v_3}{r} + \frac{\partial q_3}{\partial z} + \frac{\partial q_2}{\partial \xi} + \frac{\partial q_1}{\partial \tau} = 0, \end{aligned} \tag{13}$$

and the boundary conditions

$$\begin{aligned} & \left[\frac{1}{2}u_1^2\frac{\partial^2 v_1}{\partial r^2} + (u_2 - h_1)\frac{\partial v_1}{\partial r} + u_1\frac{\partial v_2}{\partial r} + v_3\right]|_{r=\lambda_\theta} \\ & = \frac{\partial u_3}{\partial t} - \lambda\frac{\partial u_2}{\partial \xi} + \left[\frac{\partial u_2}{\partial z} + \frac{\partial u_1}{\partial \xi}\right]q_1|_{r=\lambda_\theta} \\ & \quad + \frac{\partial u_1}{\partial z}\left[u_1\frac{\partial q_1}{\partial r} + q_2\right]|_{r=\lambda_\theta}, \\ & \left[\frac{1}{2}u_1^2\frac{\partial^2 \bar{p}_1^2}{\partial r^2} + (u_2 - h_1)\frac{\partial \bar{p_1}}{\partial r} + u_1\frac{\partial \bar{p_2}}{\partial r} + \bar{p_3}\right]|_{r=\lambda_\theta} \\ & = p_3. \end{aligned} \tag{14}$$

Here the coefficients of $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2, \beta_3$ and γ_1 are defined by

$$\begin{aligned} & \alpha_0 = \frac{1}{\lambda_\theta}\frac{\partial \Sigma}{\partial \lambda_z}, \quad \alpha_1 = \frac{1}{2\lambda_\theta}\frac{\partial^2 \Sigma}{\partial \lambda_\theta\lambda_z}, \quad \alpha_2 = \frac{1}{2\lambda_\theta}\frac{\partial^3 \Sigma}{\partial \lambda_\theta^2\lambda_z}, \\ & \gamma_1 = \frac{\lambda_z}{2\lambda_\theta}\frac{\partial^2 \Sigma}{\partial \lambda_z^2}, \quad \beta_0 = \frac{1}{\lambda_\theta\lambda_z}\frac{\partial \Sigma}{\partial \lambda_\theta}, \\ & \beta_1 = \frac{1}{\lambda_\theta\lambda_z}\frac{\partial^2 \Sigma}{\partial \lambda_\theta^2} - \frac{\beta_0}{\lambda_\theta}, \quad \beta_2 = \frac{1}{2\lambda_\theta\lambda_z}\frac{\partial^3 \Sigma}{\partial \lambda_\theta^3} - \frac{\beta_1}{\lambda_\theta}, \\ & \beta_3 = \frac{1}{6\lambda_\theta\lambda_z}\frac{\partial^4 \Sigma}{\partial \lambda_\theta^4} - \frac{\beta_2}{\lambda_\theta}. \end{aligned} \tag{15}$$

Equation (15) are defined through series expansion of the stretch ratios λ_1 and λ_2, which read

$$\begin{aligned}\lambda_1 &= \lambda_z\Bigg[1+\Bigg\{\epsilon\frac{\partial u_1}{\partial z}+\epsilon^2\left[\frac{\partial u_2}{\partial z}+\frac{\partial u_1}{\partial \xi}\right]\\&\quad+\epsilon^3\left[\frac{\partial u_3}{\partial z}+\frac{\partial u_2}{\partial \xi}+\frac{\partial u_1}{\partial \tau}\right]\Bigg\}^2\Bigg]^{1/2},\\\lambda_2 &= \lambda_\theta+\epsilon u_1+\epsilon^2[u_2-h_1(\tau)]\\&\quad+\epsilon^3[u_3-h_2(\tau)]. \qquad (16)\end{aligned}$$

By solving the sets of differential equations (9)-(14), we obtain the following nonlinear Schrödinger equation with variable coefficient :

$$i\frac{\partial U}{\partial \tau}+\mu_1\frac{\partial^2 U}{\partial \xi^2}+\mu_2|U|^2U+\mu_3 h_1 U=0, \qquad (17)$$

where

$$\begin{aligned}\mu &= \left[2\alpha_0 k+\frac{\omega^2}{k}\lambda_\theta\{f_0^2-1\}\right]^{-1},\\\mu_1 &= \mu\Bigg\{-3\lambda^2\left[\frac{f_0}{k}+\frac{m}{\lambda_\theta\lambda_z}\right]\\&\quad+\frac{\omega^2}{2k^2}\left[4\frac{\lambda_\theta\lambda k}{\omega}f_0^2-\lambda_\theta f_0^2-\lambda_\theta\right]\\&\quad+2\lambda\left[\frac{m\omega}{\lambda_z}f_0+2\frac{k}{\omega}\alpha_0\right]-\alpha_0\left[1+2k\lambda_\theta f_0\right]\Bigg\},\\\mu_2 &= \mu\Bigg\{\omega^2\Bigg[4\frac{\lambda}{\omega\lambda_\theta^2}f_0+6kf_0+\frac{3}{k\lambda_\theta^2}f_0-\frac{1}{2\lambda_\theta}\\&\quad-\frac{13}{2\lambda_\theta}f_0^2-4kf_0^2F_0+\frac{2}{\lambda_\theta}f_0F_0\Bigg]-5\frac{\alpha_1}{\lambda_\theta}k^2\\&\quad+3\left[\beta_3-\frac{\beta_1}{\lambda_\theta^2}\right]+2\alpha_2k^2-\frac{3}{2}\alpha_0k^4+3\gamma_1k^4\\&\quad+\Bigg[4\frac{\lambda\omega}{\lambda_\theta}f_0+\omega^2\left\{f_0^2-\frac{2}{k\lambda_\theta}f_0-1\right\}+2\beta_2\\&\quad+2\alpha_1k^2+\frac{\beta_1}{\lambda_\theta}\Bigg]\left[2\lambda^2-\lambda_\theta\beta_1\right]^{-1}\\&\quad\times\Bigg[4\lambda\omega f_0-2\frac{\lambda^2}{\lambda_\theta}+2\beta_1+2\alpha_1\lambda_\theta k^2+2\beta_2\lambda_\theta\\&\quad+\omega^2\left\{\lambda_\theta f_0^2-\lambda_\theta-\frac{2}{k}f_0\right\}\Bigg]\\&\quad+\Bigg[\omega^2\left\{f_0^2-\frac{6}{k\lambda_\theta}f_0-3+4f_0F_0\right\}+2\beta_2\\&\quad+6\alpha_1k^2+5\frac{\beta_1}{\lambda_\theta}\Bigg]\left[-3\beta_1+4\frac{\omega^2}{k}f_0-2\frac{\omega^2}{k}F_0\right]^{-1}\\&\quad\times\Bigg[\omega^2\left\{\frac{3}{2}-\frac{1}{2}f_0^2+\frac{1}{k\lambda_\theta}f_0\right\}-3\alpha_1k^2\\&\quad+\frac{\omega^2}{k}\left\{\frac{1}{\lambda_\theta}-2kf_0\right\}F_0-\beta_2-\frac{\beta_1}{\lambda_\theta}\Bigg]\Bigg\}\\\mu_3 &= \mu\Bigg\{-\frac{\lambda_\theta\beta_1}{2\lambda^2-\lambda_\theta\beta_1}\Bigg[\omega^2\left\{f_0^2-\frac{2}{k\lambda_\theta}f_0-1\right\}+2\beta_2\\&\quad+\frac{\beta_1}{\lambda_\theta}+4\frac{\omega\lambda}{\lambda_\theta}f_0+2\alpha_1k^2\Bigg]-2(\alpha_1k^2+\beta_2)\\&\quad+\omega^2\left\{\frac{2}{k\lambda_\theta}f_0-f_0^2+1\right\}-\frac{\beta_1}{\lambda_\theta}\Bigg\}. \qquad (18)\end{aligned}$$

The group velocity, λ is defined as

$$\lambda=\frac{\omega^2\lambda_\theta\left[f_0^2-1\right]+2\alpha_0k^2}{2\omega\left[f_0+\frac{mk}{\lambda_\theta\lambda_z}\right]}, \qquad (19)$$

and the following dispersion holds true :

$$\frac{\omega^2 f_0}{k}+\frac{m\omega^2}{\lambda_\theta\lambda_z}-\alpha_0k^2-\beta_1=0. \qquad (20)$$

Referring equation (18), we have defined the following functions as:

$$f_0=\frac{I_0(k\lambda_\theta)}{I_1(k\lambda_\theta)}, \qquad F_0=\frac{I_0(2k\lambda_\theta)}{I_1(2k\lambda_\theta)}, \qquad (21)$$

where I_n referred as modified Bessel's function of order n.

Introducing the following change of variable:

$$U=V(\xi,\tau)\exp\left[i\mu_3\int_0^\tau h_1(s)ds\right], \qquad (22)$$

equation (17) reduces to the following conventional NLS equations:

$$i\frac{\partial V}{\partial \tau}+\mu_1\frac{\partial^2 V}{\partial \xi^2}+\mu_2|V|^2V=0. \qquad (23)$$

IV. Progressive Wave Solution

In this section, we will present the progressive wave solution to the evolution equation given in (23) of the following form:

$$\begin{aligned}V(\xi,\ \tau) &= F(\zeta)\exp\left[i(K\xi-\Omega\tau)\right],\\\zeta &= \beta(\xi-2\mu_1K\tau), \qquad (24)\end{aligned}$$

where β, Ω, and K are some constants and $F(\zeta)$ is a real-valued unknown function to be determined from the solution. Introducing (24) into (23), we have

$$\mu_1\beta^2\frac{\partial^2 F}{\partial \zeta^2}+(\Omega-\mu_1K^2)F(\zeta)+\mu_2F^3(\zeta)=0. \qquad (25)$$

By solving the equation (25), we obtain the solution of NLS equation with variable coefficient (17) as

$$\begin{aligned}U(\xi,\tau) &= a\,sech\left[\sqrt{\frac{\mu_2}{2\mu_1}}a(\xi-2\mu_1K\tau)\right]\times\\&\quad\exp\left[i(K\xi-\Omega\tau+\mu_3\int_0^\tau h_1(s)ds)\right], (26)\end{aligned}$$

where $\Omega=\mu_1K^2-\frac{\mu_2}{2}a^2$.

The solution of NLS equation with variable coefficient (17) versus space τ at different time ξ is shown in Figure (1).The NLS equation with variable coefficient shows bell-shape wave propagates to the left as time, ξ passed.

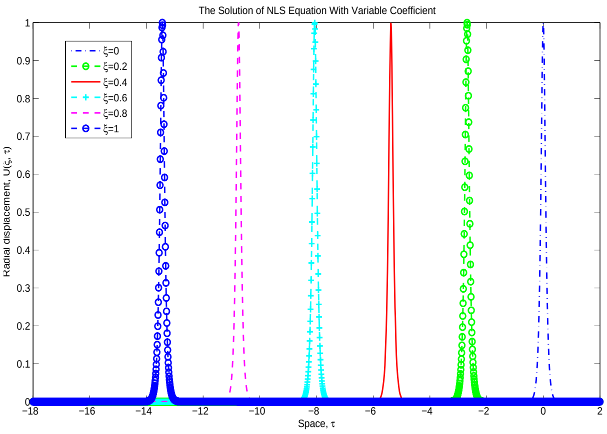

Fig. 1. The solutio of NLS equation with variable coefficient versus space τ

V. Conclusion

It has been presented the modulation of nonlinear waves in a prestressed thin-walled elastic tube with a symmetrical stenosis filled with a full inviscid fluid. The governing equation is obtained as the nonlinear Schrödinger (NLS) equation with variable coefficient.

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